

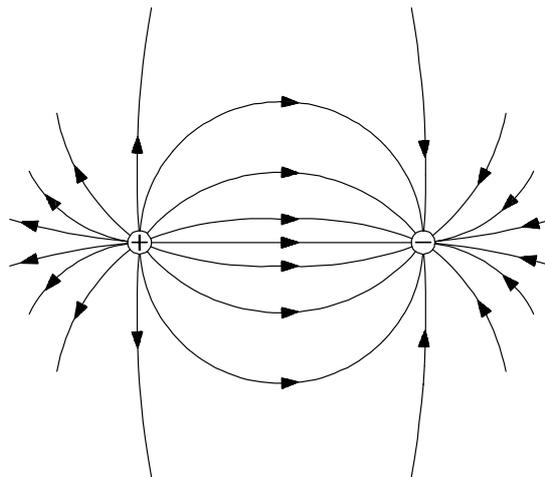
Gauss's Law

The electric field and field lines

Electric field lines provide a convenient way of visualising the electric field.

The field lines are drawn according to the following rules:

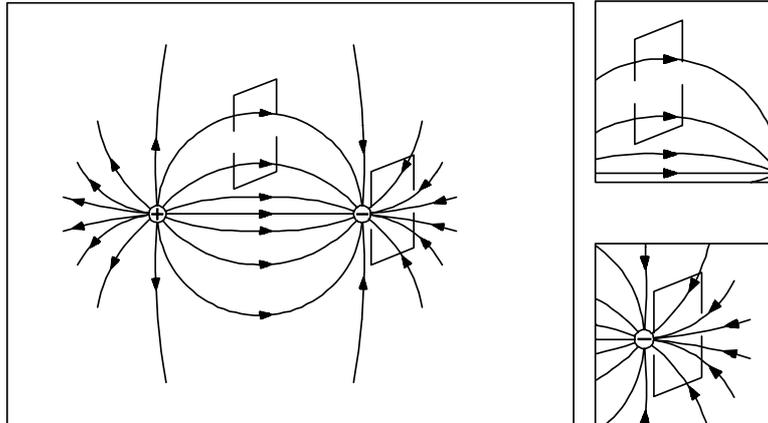
1. The field lines point in the same direction as the electric field at every point in space. Therefore the electric field at a point is always at a tangent to the field lines at that point.
2. Field lines always start on the positive charge and end on the negative charge.
3. The strength of the field is represented by the density of the lines. The number of lines, in 3 dimensions, leaving a positive charge or approaching a negative charge is proportional to the magnitude of the charge.
4. No two lines touch or cross.



Electric Flux

Definition of flux: The flux of any vector quantity through an area is the product of the area and the component of the vector at right angles to the area.

For an electric field, the flux ϕ is measured by the number of lines of force that pass through a hypothetical surface.



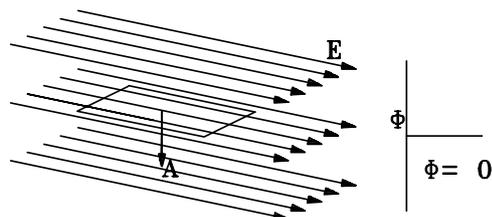
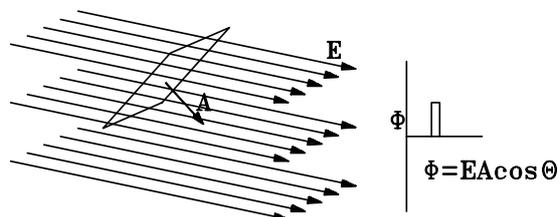
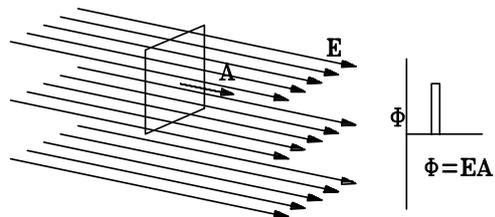
For a flat surface in a uniform electric field \mathbf{E} the electric flux is:

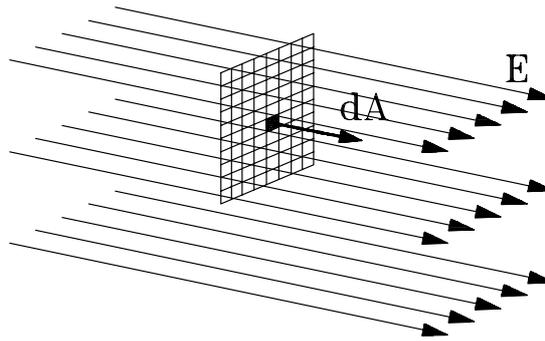
$$\phi = EA \cos \theta$$

where: ϕ is the flux through the surface A
 A is the surface area
 θ is the angle between the surface vector and the field vector

This can be generalised for any surface by:

$$\Phi = \int_{\text{surface}} \mathbf{E} \cdot d\mathbf{A}$$





If a closed surface (such as a sphere or cube etc.) encloses no net charge then its total flux is zero.

The total flux through a surface enclosing a net charge is proportional to the value of the charge.

Gaussian surfaces

A Gaussian surface is an imaginary surface drawn so as to be able to use Gauss's law.

$d\mathbf{A}$ represents a small area of the surface and is known as the surface vector. The vectors magnitude is the small surface area; its direction is represented as being normal to the surface and away from the volume.

The surface must surround some or all of the charge and must go through the point where the electric field is to be calculated.

The surface is chosen to represent the symmetry of the system.

Gauss's Law

The total electric flux of a closed surface in an electric field is 4π times the electric charge within that surface.

or

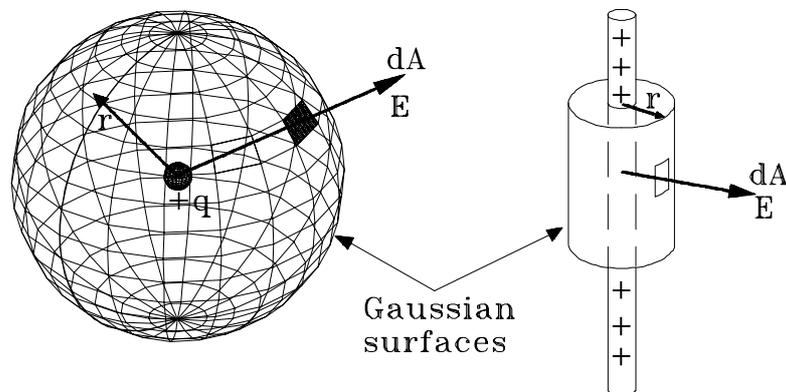
The total electric flux through a closed surface is proportional to the charge enclosed by that surface.

$$\phi = q_n/\epsilon_0$$

$$q_n = \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{A}$$

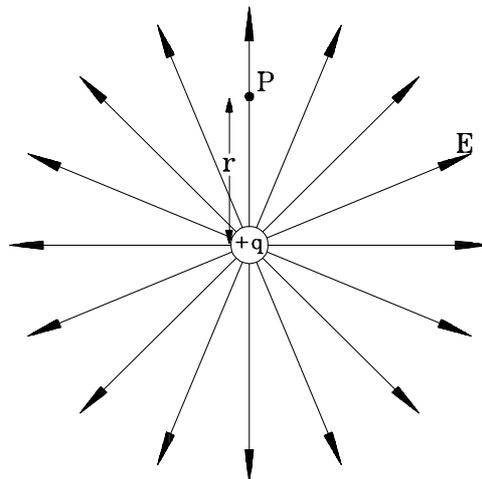
where: q_n is the net charge enclosed by the surface
 $\mathbf{E} \cdot d\mathbf{A}$ is the flux through a small surface element
 $\oint \mathbf{E} \cdot d\mathbf{A}$ is the total flux through a closed surface

This integral can be difficult to solve unless the surface chosen exploits the symmetry of a system which can simplify the solution.



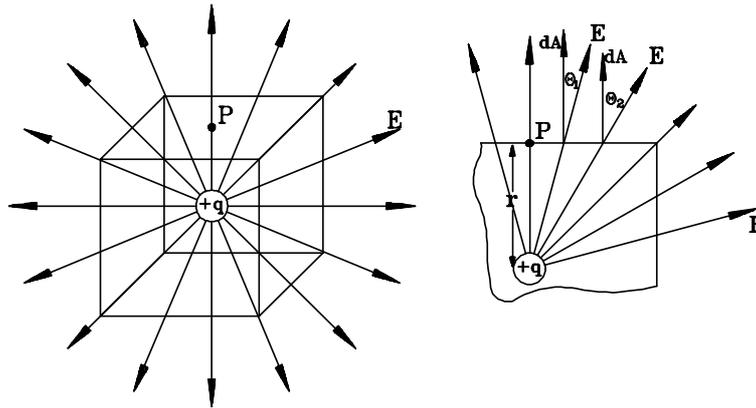
Application to a point charge (zero dimension)

It is required to calculate the electric field at a point P due to a single point charge.



First choose a closed gaussian surface which encloses the charge and passes through the point P.

To illustrate the difficulties which can be encountered if we do not follow the general rule of selecting a surface to mirror the symmetry of the system, we will first choose a cube as the Gaussian surface.



It can be seen that the integral will be extremely difficult to perform as the flux lines pass through the surfaces at varying angles i.e. the angle between the surface vector and the electric field \mathbf{E} changes.

By looking at the symmetry of the system we can select a sphere as the Gaussian surface. The electric field vectors are now always parallel to the surface vectors and therefore the electric field has the same magnitude over the complete surface.

$$q_n = \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{A}$$

$$q_n = \epsilon_0 E 4\pi r^2$$

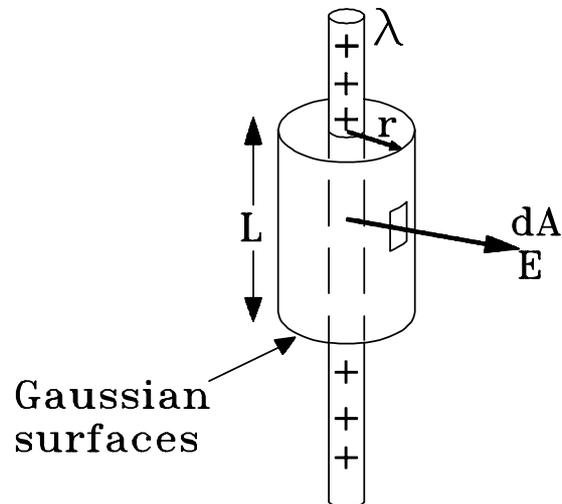
$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

and

$$\Phi = \frac{q}{\epsilon_0} = 4\pi r^2 E$$

Uniform line of charge (1 dimension)

We can calculate the electric field created by an infinitely long line of charge, at any point in space, by using a cylindrical Gaussian surface.



The net charge enclosed by the surface = λL

The electric field lines are parallel to the surface vectors of the round surface of the cylinder and perpendicular to the surface vectors of the top and bottom.

Perform the integration for each surface:

$$\frac{q}{\epsilon_0} = \int_{Top} E dA + \int_{Bottom} E dA + \int_{Side} E dA$$

$$\frac{q}{\epsilon_0} = \int_{Top} E dA \cos 90 + \int_{Bottom} E dA \cos 90 + \int_{Side} E dA \cos 0$$

$$\frac{q}{\epsilon_0} = \int_{Side} E dA \cos 0$$

$$\frac{q}{\epsilon_0} = E \int dA = E 2\pi r L$$

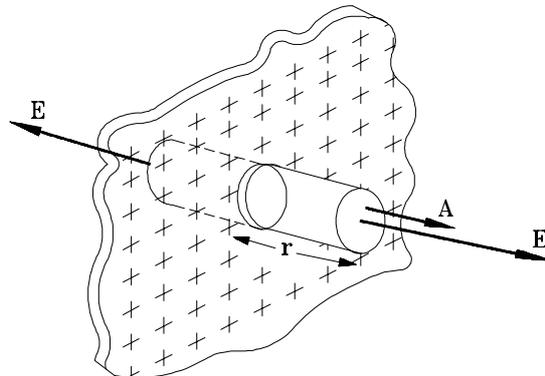
This can be rearranged to give the magnitude of the electric field:

$$E = \frac{q}{L} \frac{1}{2\pi\epsilon_0 r} = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r}$$

The electric field from an infinitely long line of charge falls as $1/r$.

A uniform sheet of charge (2 dimension)

For a uniform sheet of charge with a charge density of σ (charge per unit area) it is useful to be able to calculate the field \mathbf{E} at any distance r from the sheet.



A convenient Gaussian surface is a cylinder of cross section A and length $2r$. This pierces the plane so that its length is perpendicular to it.

\mathbf{E} points perpendicularly away from the plane in both directions and is therefore parallel to the end cap surface vectors. \mathbf{E} is always perpendicular to the curved side of the cylinder and therefore, as previously seen, can be ignored.

$$\frac{q}{\epsilon_0} = \int_{Top} \mathbf{E} d\mathbf{A} + \int_{Bottom} \mathbf{E} d\mathbf{A} + \int_{Side} \mathbf{E} d\mathbf{A}$$

$$\frac{q}{\epsilon_0} = \int_{Top} \mathbf{E} d\mathbf{A} + \int_{Bottom} \mathbf{E} d\mathbf{A}$$

$$\frac{q}{\epsilon_0} = E(A + A) = 2EA$$

Substituting the charge density into this equation gives:

$$E = \frac{\sigma}{2\epsilon_0}$$

Therefore E is the same magnitude for all points on each side of the plain.

Spherical symmetric charge distribution (3 dimension)

For a spherical charge distribution of radius R, the charge density ρ (charge per unit volume) at any point depends only on the distance of the point from the centre and not on the direction - this is called spherical symmetry. There are 2 cases to be considered:

1. To find E for points outside the charge sphere, we assign a Gaussian spherical surface of radius r where $r > R$. Applying Gauss's law:

$$q = \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{A}$$

$$q = \epsilon_0 E 4\pi r^2$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad \text{q is the total charge}$$

This is the same result as obtained for a point charge. Therefore it can be said that for points outside a spherical symmetric distribution of charge, the electric field has a value that it would have if the charge were concentrated at its centre.

2. To find E for points inside the charge sphere, we assign a Gaussian spherical surface of radius r where $r < R$.

$$q^* = \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{A}$$

$$q^* = \epsilon_0 E 4\pi r^2$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{q^*}{r^2} \quad \text{where } q^* \text{ is the part of } q \text{ contained within the sphere of radius } r$$

If the charge is uniformly distributed and ρ would then have a constant value for all points within the sphere of radius R, and is zero outside of the sphere.

For points inside the sphere, then

$$q^* = q \frac{\frac{4}{3} \pi r^3}{\frac{4}{3} \pi R^3}$$

$$q^* = q \left(\frac{r}{R} \right)^3$$

Then the expression for E becomes:

$$E = \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3}$$

Relationship between distance r and dimensionality

An electric field's dependence on distance r depends on the dimensionality of the charge distribution that creates it.

0-dimension: the electric field for point charge falls as $1/r^2$

1-dimension: the electric field for a line of charge falls as $1/r$

2-dimension: the electric field for plane of charge is constant everywhere in space

3-dimension: the electric field for a spherical distribution of charge rises with r

From the above it can be seen that:

$$\text{THE ELECTRIC FIELD CHANGES AS } r^{D-2}$$

Where D is the dimensionality of the charge distribution.

