Differential Equations

DIRECT INTEGRATION

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A Tutorial Module introducing ordinary differential equations and the method of direct integration

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1. Introduction

$$\boxed{\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = x^7}$$ is an example of an **ordinary** differential equation (o.d.e.) since it contains only **ordinary** derivatives such as $\frac{dy}{dx}$ and not **partial** derivatives such as $\frac{\partial y}{\partial x}$.

The dependent variable is y while the independent variable is x (an o.d.e. has only one independent variable while a partial differential equation has more than one independent variable).

The above example is a **second order** equation since the highest order of derivative involved is **two** (note the presence of the $\frac{d^2y}{dx^2}$ term).

An o.d.e. is **linear** when each term has y and its derivatives only appearing to the power one. The appearance of a term involving the product of y and $\frac{dy}{dx}$ would also make an o.d.e. **nonlinear**.

In the above example, the term $\left(\frac{dy}{dx}\right)^3$ makes the equation **nonlinear**.

The **general solution** of an n^{th} order o.d.e. has $\boldsymbol{n}$ arbitrary constants that can take any values.

In an **initial value problem**, one solves an $\boldsymbol{n^{th}}$ order o.d.e. to find the general solution and then applies $\boldsymbol{n}$ **boundary conditions** ("initial values/conditions") to find a **particular solution** that does not have any arbitrary constants.

2. Theory

An ordinary differential equation of the following form:

$$\frac{dy}{dx} = f(x)$$

can be solved by integrating both sides with respect to x:

$$y = \int f(x)\,dx\ .$$

This technique, called **DIRECT INTEGRATION**, can also be applied when the left hand side is a higher order derivative.

In this case, one integrates the equation a sufficient number of times until y is found.

3. Exercises

Click on EXERCISE links for full worked solutions (there are 8 exercises in total)

EXERCISE 1.

Show that $y = 2e^{2x}$ is a particular solution of the ordinary differential equation: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

EXERCISE 2.

Show that $y = 7\cos 3x - 2\sin 2x$ is a particular solution of $\frac{d^2y}{dx^2} + 2y = -49\cos 3x + 4\sin 2x$

Exercise 3.

Show that $y = A\sin x + B\cos x$, where A and B are arbitrary constants, is the general solution of $\frac{d^2y}{dx^2} + y = 0$

Exercise 4.

Derive the general solution of $\frac{dy}{dx} = 2x + 3$

Exercise 5.

Derive the general solution of $\frac{d^2y}{dx^2} = -\sin x$

Exercise 6.

Derive the general solution of $\frac{d^2y}{dt^2} = a$, where $a = \text{constant}$

Exercise 7.

Derive the general solution of $\dfrac{d^3y}{dx^3} = 3x^2$

Exercise 8.

Derive the general solution of $e^{-x}\dfrac{d^2y}{dx^2} = 3$

4. Answers

1. HINT: Work out $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ and then substitute your results, along with the given form of y, into the differential equation ,

2. HINT: Show that $\frac{d^2y}{dx^2} = -63\cos 3x + 8\sin 2x$ and substitute this, along with the given form of y, into the differential equation ,

3. HINT: Show that $\frac{d^2y}{dx^2} = -A\sin x - B\cos x$,

4. $y = x^2 + 3x + C$,

5. $y = \sin x + Ax + B$,

6. $y = \frac{1}{2}at^2 + Ct + D$,

7. $y = \frac{1}{20}x^5 + C'x^2 + Dx + E$,

8. $y = 3e^x + Cx + D$.

5. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\|x\|$	$\frac{g'(x)}{g(x)}$	$\ln\|g(x)\|$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\|\cos x\|$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\|\tan\frac{x}{2}\right\|$	$\operatorname{cosech} x$	$\ln\left\|\tanh\frac{x}{2}\right\|$
$\sec x$	$\ln\|\sec x + \tan x\|$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\|\sin x\|$	$\coth x$	$\ln\|\sinh x\|$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\|\ (0<\|x\|<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\|\ (\|x\|>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\|\frac{x+\sqrt{a^2+x^2}}{a}\right\|\ (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\|\frac{x+\sqrt{x^2-a^2}}{a}\right\|\ (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

6. Tips on using solutions

● When looking at the THEORY, ANSWERS, INTEGRALS or TIPS pages, use the Back button (at the bottom of the page) to return to the exercises.

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct.

● Try to make less use of the full solutions as you work your way through the Tutorial.

Full worked solutions

Exercise 1.

We need:

$$
\begin{aligned}
\frac{dy}{dx} &= 2 \cdot 2e^{2x} = 4e^{2x} \\
\frac{d^2y}{dx^2} &= 2 \cdot 4e^{2x} = 8e^{2x}. \\
\therefore \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y &= 8e^{2x} - 4e^{2x} - 2 \cdot e^{2x} \\
&= (8-8)e^{2x} \\
&= 0 \\
&= \text{RHS}
\end{aligned}
$$

Return to Exercise 1

Exercise 2.

To show that $y = 7\cos 3x - 2\sin 2x$ is a particular solution of $\frac{d^2y}{dx^2} + 2y = -49\cos 3x + 4\sin 2x$, work out the following:

$$\begin{aligned}\frac{dy}{dx} &= -21\sin 3x - 4\cos 2x\\ \frac{d^2y}{dx^2} &= -63\cos 3x + 8\sin 2x\end{aligned}$$

$$\begin{aligned}\therefore\quad \frac{d^2y}{dx^2} + 2y &= -63\cos 3x + 8\sin 2x + 2(7\cos 3x - 2\sin 2x)\\ &= (-63+14)\cos 3x + (8-4)\sin 2x\\ &= -49\cos 3x + 4\sin 2x\\ &= \text{RHS}\end{aligned}$$

Notes
- The equation is second order, so the general solution would have two arbitrary (undetermined) constants.
- Notice how similar the particular solution is to the Right-Hand-Side of the equation. It involves the same functions but they have different coefficients i.e.

y is of the form

$$\text{“}a\cos 3x + b\sin 2x\text{”} \qquad \begin{pmatrix} a = 7 \\ b = -2 \end{pmatrix}$$

Return to Exercise 2

Exercise 3.

We need:

$$
\begin{aligned}
\frac{dy}{dx} &= A\cos x + B\cdot(-\sin x) \\
\frac{d^2y}{dx^2} &= -A\sin x - B\cos x \\
\therefore \quad \frac{d^2y}{dx^2} + y &= (-A\sin x - B\cos x) + (A\sin x + B\cos x) \\
&= 0 \\
&= \text{RHS}
\end{aligned}
$$

Since the differential equation is second order and the solution has two arbitrary constants, this solution is the general solution.

Return to Exercise 3

Exercise 4.

This is an equation of the form $\frac{dy}{dx} = f(x)$, and it can be solved by direct integration.

Integrate both sides with respect to x:

$$\begin{aligned} \int \frac{dy}{dx} dx &= \int (2x+3) dx \\ \text{i.e.} \quad \int dy &= \int (2x+3) dx \\ \text{i.e.} \quad y &= 2 \cdot \frac{1}{2} x^2 + 3x + C \\ \text{i.e.} \quad y &= x^2 + 3x + C, \end{aligned}$$

where C is the (combined) arbitrary constant that results from integrating both sides of the equation. The general solution must have one arbitrary constant since the differential equation is first order.

Return to Exercise 4

Exercise 5.

This is of the form $\boxed{\frac{d^2y}{dx^2} = f(x)}$, so we can solve for y by direct integration.

Integrate both sides with respect to x:

$$\begin{aligned} \frac{dy}{dx} &= -\int \sin x dx \\ &= -(-\cos x) + A \end{aligned}$$

Integrate again:

$$y = \sin x + Ax + B$$

where A, B are the two arbitrary constants of the general solution (the equation is second order).

Return to Exercise 5

Exercise 6.

Integrate both sides with respect to t:

$$\begin{aligned} \frac{dy}{dt} &= \int a\,dt \\ \text{i.e. } \frac{dy}{dt} &= at + C \end{aligned}$$

Integrate again:

$$\begin{aligned} y &= \int (at + C)dt \\ \text{i.e. } y &= \frac{1}{2}at^2 + Ct + D\,, \end{aligned}$$

where C and D are the two arbitrary constants required for the general solution of the second order differential equation.

Return to Exercise 6

Exercise 7.

Integrate both sides with respect to x:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \int 3x^2dx \\ \text{i.e. } \frac{d^2y}{dx^2} &= 3\cdot\frac{1}{3}x^3 + C \\ \text{i.e. } \frac{d^2y}{dx^2} &= x^3 + C\end{aligned}$$

Integrate again:

$$\begin{aligned}\frac{dy}{dx} &= \int (x^3 + C)dx \\ \text{i.e. } \quad \frac{dy}{dx} &= \frac{x^4}{4} + Cx + D\end{aligned}$$

Integrate again:

$$y = \int \left(\frac{x^4}{4} + Cx + D \right) dx$$

$$\text{i.e.} \quad y = \frac{1}{4} \cdot \frac{1}{5} x^5 + C \cdot \frac{1}{2} x^2 + Dx + E$$

$$\text{i.e.} \quad y = \frac{1}{20} x^5 + C' x^2 + Dx + E$$

where $C' = \frac{C}{2}$, D and E are the required three arbitrary constants of the general solution of the third order differential equation.

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Exercise 8.

Multiplying both sides of the equation by e^x gives:

$$e^x \cdot e^{-x}\frac{d^2y}{dx^2} = e^x \cdot 3$$

$$\text{i.e.} \quad \frac{d^2y}{dx^2} = 3e^x$$

This is now of the form $\frac{d^2y}{dx^2} = f(x)$, where $f(x) = 3e^x$, and the solution y can be found by direct integration.

Integrating both sides with respect to x:

$$\frac{dy}{dx} = \int 3e^x dx$$

$$\text{i.e.} \quad \frac{dy}{dx} = 3e^x + C\,.$$

Integrate again:

$$y = \int (3e^x + C)dx$$

$$\text{i.e.} \quad y = 3e^x + Cx + D\,,$$

where C and D are the two arbitrary constants of the general solution of the original second order differential equation.

Return to Exercise 8