Flexural waves in incompressible pre-stressed elastic composites

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Abstract

In this paper some dynamic properties of symmetric, 4-ply (3 layer), pre-stressed, incompressible elastic laminated plates are discussed. The dispersion relation arising from flexural wave propagation is derived in respect of arbitrary strain energy functions. An asymptotic high wave number analysis is carried out which establishes that there are six possible phase speed limits of the dispersion relation, these being a surface wave speed, an interfacial wave speed or one of four associated shear wave speeds. All harmonics, with the possible exception of the first, are shown to tend to the least of all four shear wave speeds. These four distinct cases are all fully analysed with asymptotic expansions for the phase speed obtained up to and including third order. In two of the four cases sinusoidal terms are found to occur in these expansions at third order. If surface and interfacial waves exist then the corresponding wave speeds are shown to be the high wave number limit of the first two branches of the dispersion relation, provided they are lower in magnitude than all four shear wave speeds. In the case of interfacial waves this will always be the case, however it is quite possible to have a surface wave speed greater than one, or even two, of these shear wave speeds. In such cases a distinct flattening of dispersion curve branches in the neighbourhood of the surface wave speed is observed, together with extremely flat maxima in neighbouring wave number regions on the associated group velocity curves. The implication is that in such cases a surface wave front will be formed from the combination of the various harmonics, rather than the high wave number limit of a single branch. The paper is concluded with a discussion of quasi-static solutions, neutral curves and stability. It is shown that possible regions exist in the low wave number regime within which bifurcation from the underlying homogeneous deformation is not possible.

1 Introduction

Problems involving vibrations and waves in linear elastic half spaces, plates and laminates have been areas of active research for the greater part of this century. The first attempts to elucidate the effect of pre-stress within this area was seemingly in the context of surface waves, see e.g. Hayes and Rivlin [1], Flavin [2] and a series of papers by Willson [3, 4, 5]. More recent contributions, also within the context of surface waves, have been carried out by Chadwick and Jarvis [6] and Dowaikh and Ogden [7]. The former authors also discussed the problem of interfacial waves propagating along the boundary of two perfectly bonded neo-Hookean materials, see Chadwick and Jarvis [8, 9]. This problem was later generalised, in the case of plane strain, to interfacial waves propagating along the boundary between two perfectly bonded incompressible elastic half spaces of arbitrary strain energy function by Dowaikh and Ogden [10]. Other relevant problems which give rise to possible interfacial wave solutions have recently been examined by Ogden and Sotiropoulos [11] and Sotiropoulos and Sifniotopoulos [12]. The first of these papers addresses the problem of an incompressible layer of finite thickness perfectly bonded to a half space, whilst the second concerns the problem of an incompressible layer sandwiched between two perfectly bonded incompressible half spaces. A further recent related contribution is a study of interfacial and surface waves in both pre-strained restricted Hadamard and neo-Hookean materials by Chadwick [13].

In this present paper the problem of flexural wave propagation in a symmetric, incompressible, pre-stressed, 4-ply (3 layer) laminated plate is discussed. At this stage it is perhaps worth noting that although such a structure consists of only three layers, it is usually referred to as 4-ply. This terminology is used because a 4-ply laminate is usually formed by bonding together two identical unit cells, these both being 2-ply. It should also be emphasised that although pre-stress is often induced in the manufacturing process, the type of scenario envisaged is one in which it arises through the action of external forces. In particular we cite the increasing use of rubber like components as vibration insulators in bridge supports and large buildings, this having specific relevance to earthquake protection, see Sheridan et al. [14]. A discussion of some preliminary results for the corresponding extensional wave problem may be found in the recent papers by Rogerson [15] and Rogerson and Sandiford [16]. However, these earlier studies are only concerned with a specific class of incompressible material. It has recently been established by one of the present authors, in the case of a single plate, that many new features arise for more general materials which could not possibly occur in the previous studies, see Rogerson [17]. It is with this in mind that we embark on a more general analysis applicable to laminated plates. Amongst papers most specifically relevant to the present study we cite the papers by Ogden and Roxburgh [18], Rogerson and Fu [19], Green [20] and Baylis and Green [21]. The first two of these papers concern vibration and wave propagation in an incompressible, pre-stressed elastic plate, the first focusing on the problems of vibration and stability, whilst the second concerns the derivation of some asymptotic expansions of the dispersion relation. The latter two papers concern bending waves in a plate and 4-ply laminated plate, respectively, each layer being composed of a linear transversely isotropic elastic solid.

We begin this paper in section 2 with a brief review of the governing equations and then derive the appropriate propagator matrix. The dispersion relation is derived in section 3 and asymptotically investigated in section 4. The long wavelength limiting results are shown to be similar to earlier related studies, the fundamental mode being the only one retaining finite wave speed. Also, the short wavelength limit of the fundamental mode is shown to be the least of a Rayleigh surface wave speed, a Stoneley interfacial wave speed or one of two so-called limiting wave speeds. These two limiting wave speeds are each associated with either the outer layers or inner core, respectively. A significant difference from earlier restricted studies is that the limiting speed in each layer is now the lower of two associated shear wave speeds, the limiting speed in each layer therefore being dependent on material parameters. The high wave number limit of all harmonics is in general the lower of the two limiting wave speeds, a possible exception when the first harmonic may tend to a surface or interfacial wave speed limit is noted. The implication is that in general there are four possible high wave number limits for the harmonics. These four possible cases are each discussed in turn and appropriate asymptotic expansion of the dispersion relation derived in each case up to and including third order. In the first two cases the results are what one would expect in light of earlier studies, however in cases 3 and 4 the character of the dispersion relation differs radically from previous studies. In these two cases the harmonics in both the moderate and high wave number regimes exhibit distinct oscillatory behaviour and appropriate sinusoidal terms are found at third order in the asymptotic solutions. A further striking feature in the final case is that the harmonics are observed to form distinct pairs. Indeed, the appropriate third order correction terms of the asymptotic expansions are shown to satisfy a quadratic equation, this then yielding an approximation to both branches in each pair, the two branches having the same approximation to second order.

In section 5 numerical results are presented which in particular focuses on the latter two cases. The asymptotic solutions are compared with the numerical solution at both second and third order and provide excellent approximations. A further point of interest is observed in the numerical solution in a case when a surface wave speed exists which is greater than the limiting wave speed in the core. In this case the high wave number limit of the fundamental mode is this limiting wave speed. However, a clear surface wave front is observed to be formed by the cumulative effect of the harmonics, manifesting itself through the flattening of the dispersion curves as each branch passes through the surface wave speed. This phenomenon is further illustrated on the corresponding group velocity curves in which distinct and extremely flat maxima are observed in adjacent wave number regimes.

A great deal of research effort has, in recent years, been focused on determination of the transient response of plates and laminates to surface and internal loads. One method of solution is to use an integral transform formulism and then obtain the transient response by numerical inversion. For a line impact it has been shown that the denominator of the solution integrand is the associated dispersion relation, see Rogerson [22]. The transient response is then obtained by summing the integral contributions from the infinite number of dispersion curve branches. A detailed knowledge of the dispersion relation is therefore crucial before this method may be employed. In addition, the asymptotic expansions derived in this paper may well help in estimating the error involved in truncating the numerical inversion at some finite wave number.

The paper is concluded in section 6 with a discussion of quasi-static solutions, neutral curves and stability. The neutral curves correspond to the vanishing of the phase speed and define the range of primary deformation for which the phase speed remains real. These curves may therefore be thought of as bifurcation conditions. Two such conditions are obtained which give the principal Cauchy stress normal to the surface, σ_2, as a function of scaled wave number. It is shown that whenever long wavelength limiting values of σ_2 exist they are finite, contrasting with the corresponding case for extensional waves for which only one value of σ_2 is finite, see Rogerson and Sandiford [16]. However, again in contrast to the extensional case, there is the possible existence of a region in the low wave number regime within which no real solutions for σ_2 exist and the phase speed always remains real. Bifurcation from the homogeneous deformation is therefore not possible within such regions. This phenomenon is discussed in further detail in the case of both materials being of neo-Hookean type with simple primary deformations.

2 Basic equations and the propagator matrix

In this section we first summarise the basic equations which govern small amplitude, time-dependent motions superimposed upon a large static primary deformation, under the assumptions of incompressible plane strain elasticity, and then derive the appropriate propagator matrix. For details concerning the derivation of any of the basic equations, many of which are merely stated, the reader is referred to the papers by Dowaikh and Ogden [7] and Rogerson and Fu [19]. The position vector of a representative particle is denoted by X_A in an initial unstressed configuration B_u, $x_i(X_A)$ in a pre-stressed equilibrium state B_e and $\tilde{x}_i(X_A, t)$ in the final time-dependent configuration B_t, thus

$$\tilde{x}_i(X_A, t) = x_i(X_A) + u_i(X_A, t) \ , \tag{2.1}$$

where $\mathbf{u}(\mathbf{x}, t)$ is a small time-dependent displacement superimposed on B_e. An appropriate Cartesian co-ordinate system $Ox_1x_2x_3$ is chosen which is coincident with the principal axes of stress in B_e. A plane strain simplification of the equations of motion yields the two non-trivial equations

$$B_{1111}u_{1,11} + (B_{1122} + B_{2112})\, u_{2,21} + B_{2121}u_{1,22} - p^*_{,1} = \rho\ddot{u}_1 \ , \tag{2.2}$$

$$(B_{1221} + B_{2211})\, u_{1,12} + B_{1212}u_{2,11} + B_{2222}u_{2,22} - p^*_{,2} = \rho\ddot{u}_2 \ , \tag{2.3}$$

in which it has been assumed $u_3 \equiv 0$ and u_1 and u_2 are independent of x_3. Furthermore, in equations (2.2) and (2.3) B_{ijkl} are components of the appropriate fourth order elasticity tensor, p^* is a time-dependent pressure increment, ρ the density of the material and a comma indicates differentiation with respect to the implied spatial co-ordinate component in B_e. In addition, two non-zero linearised traction increments are obtainable in the component form

$$\tau_1 = B_{2121} u_{1,2} + (B_{2112} + \bar{p})\, u_{2,1} \ , \tag{2.4}$$

$$\tau_2 = B_{2211} u_{1,1} + (B_{2222} + \bar{p})\, u_{2,2} - p^* \ , \tag{2.5}$$

with $\bar{p}$ denoting a static pressure in B_e.

As our ultimate aim is to consider laminates which are finite in the Ox_2 direction, solutions of equations (2.2) and (2.3) are now sought in the form of the travelling wave

$$(u_1, u_2, p^*) = (U, V, kP)\, e^{kqx_2} e^{ik(x_1 - vt)} \ , \tag{2.6}$$

in which k is the wave number, v the phase speed and q is to be determined. Insertion of equation (2.6) into (2.2) and (2.3), in conjunction with use of the reduced linearised incompressibility condition, yields a system of homogeneous equations which admit a non-trivial solution provided

$$\gamma q^4 + \left(\rho v^2 - 2\beta\right) q^2 + \left(\alpha - \rho v^2\right) = 0 \ , \tag{2.7}$$

within which

$$\alpha = B_{1212} \ , \ 2\beta = B_{1111} + B_{2222} - 2B_{1122} - 2B_{1221} \ , \ \gamma = B_{2121} \ .$$

For use in later sections it is remarked that if the two roots of (2.7) are denoted by q_1^2 and q_2^2 then

$$q_1^2 + q_2^2 = \frac{2\beta - \rho v^2}{\gamma}, \quad q_1^2 q_2^2 = \frac{\alpha - \rho v^2}{\gamma}. \tag{2.8}$$

Solutions for U, V and P are now obtainable as linear combinations of the four solutions obtained from (2.7). These solutions may be used with (2.4) and (2.5) to establish that

$$\mathbf{Y}(x_2) = \mathbf{H}\mathbf{E}(x_2)\mathbf{A} \ , \tag{2.9}$$

where $\mathbf{Y}(x_2)$ is a displacement-traction four vector and $\mathbf{E}(x_2)$ a diagonal matrix given by

$$\mathbf{Y}(x_2) = \left(-iU, V, \frac{\tau_1}{ik}, \frac{\tau_2}{k}\right)^T \ , \qquad \mathbf{E}(x_2) = diag\left(E_1^+, E_1^-, E_2^+, E_2^-\right) \ ,$$

$\mathbf{A}$ is a vector of arbitrary constants, $E_m^+ = e^{kq_m x_2}$, $E_m^- = e^{-kq_m x_2}$, $\mathbf{H}$ is the 4×4 matrix

$$\mathbf{H} = \begin{bmatrix} q_1 & -q_1 & q_2 & -q_2 \\ 1 & 1 & 1 & 1 \\ f(q_1) & f(q_1) & f(q_2) & f(q_2) \\ q_1 f(q_2) & -q_1 f q_2) & q_2 f(q_1) & -q_2 f(q_1) \end{bmatrix} ,$$

and $f(q) = \gamma(1+q^2) - \sigma_2$. It is perhaps worth remarking at this stage that although σ_2, the principal Cauchy stress along Ox_2 in B_e, appears in the governing equations explicitly, this is only one of several possible representations. The implication is that the pre-stress considered is the most general arising from a pure homogeneous strain. The arbitrary constant vector $\mathbf{A}$ may be eliminated from the solution shown in (2.9) in favour of the unknown values of $\mathbf{Y}(\bar{x}_2)$ at some specific location $x_2 = \bar{x}_2$ to yield

$$\mathbf{Y}(x_2) = \mathbf{H}\mathbf{E}(x_2)\mathbf{E}^{-1}(\bar{x}_2)\mathbf{H}^{-1}\mathbf{Y}(\bar{x}_2) = \mathbf{P}(x_2 - \bar{x}_2)\mathbf{Y}(\bar{x}_2) \ . \tag{2.10}$$

In (2.10) $\mathbf{P}(x_2 - \bar{x}_2)$ is the so called propagator matrix, see Gilbert and Backus [23], is defined by

$$\mathbf{P}(x_2 - \bar{x}_2) = \mathbf{H}\mathbf{E}(x_2 - \bar{x}_2)\mathbf{H}^{-1} \ . \tag{2.11}$$

For specified values of the material constants ρ, α, β and γ the propagator matrix is a function of wave number k, phase speed v and the distance $x_2 - \bar{x}_2 = h$. The elements of the propagator matrix $\mathbf{P}(h)$ have been calculated and are given explicitly in appendix A at the end of this paper.

3 Derivation of the Dispersion Relation

Consider now a 4-ply laminated plate which is symmetrical about its mid-plane, consists of an inner core of thickness $2d$, two identical outer layers of thickness h and is infinite in each of the other two spatial directions. The material of both the inner core and the outer layers is that of a pre-stressed, incompressible elastic solid with principal axes of stress assumed coincident for all layers of the laminate. An appropriate Cartesian coordinate system $Ox_1x_2x_3$ is chosen which is coincident with the principal axes of stress in the pre-stressed equilibrium state B_e, with Ox_2 normal to the plane of the plate and the origin O in the mid-plane. The inner core of the plate then occupies the region $-d \leq x_2 \leq d$, with the upper and lower layers occupying $d \leq x_2 \leq (d+h)$ and $-(d+h) \leq x_2 \leq -d$, respectively.

The material parameters and density for the outer layers of the laminate are denoted by α, β, γ and ρ, as defined immediately after equation (2.7). The corresponding parameters for the inner core are denoted by $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ and $\tilde{\rho}$ and lead to different solutions of the governing equations. Accordingly, appropriate solutions of equations (2.2) and (2.3) for the core are now sought in the form

$$(u_1, u_2, p^*) = \left(\tilde{U}, \tilde{V}, k\tilde{P}\right) e^{kpx_2} e^{ik(x_1 - vt)} \,. \tag{3.1}$$

On inserting equation (3.1) into the governing equations (2.2)–(2.5) and following the preceding analysis we deduce that

$$\tilde{\mathbf{Y}}(x_2) = \tilde{\mathbf{P}}(x_2 - \bar{x}_2)\tilde{\mathbf{Y}}(\bar{x}_2) \,, \tag{3.2}$$

where $\tilde{\mathbf{Y}}(x_2)$ is the displacement-traction four vector for the inner core, which is defined by $\tilde{\mathbf{Y}} = \left(-i\tilde{U}, \tilde{V}, \frac{\tilde{\tau}_1}{ik}, \frac{\tilde{\tau}_2}{k}\right)^T$ and $\tilde{\mathbf{P}}(x_2 - \bar{x}_2)$ is the appropriate propagator matrix. For the distance $x_2 - \bar{x}_2 = d$,the elements of the propagator matrix $\tilde{\mathbf{P}}$ may be deduced directly from $\mathbf{P}$, with obvious notation, by the interchanging of $q_m \leftrightarrow p_m$,$f(q_m) \leftrightarrow \tilde{f}(p_m)$, $C_m \leftrightarrow \tilde{C}_m$ and $S_m \leftrightarrow \tilde{S}_m$, where $\tilde{C}_m = \operatorname{Cosh} kp_m d$ and $\tilde{S}_m = \operatorname{Sinh} kp_m d$. Due to the symmetrical lay up of the laminate the solutions of the governing equations in the lowermost layer are obtainable from (2.9) by making appropriate sign changes to the column vector $\mathbf{A}$. It is also noted that in order to ensure continuity at the perfectly bonded interfaces it will be assumed that $\sigma_2 = \tilde{\sigma}_2$. The continuity condition at the upper perfectly bonded interface may now be expressed in the form

$$\mathbf{Y}(d) = \tilde{\mathbf{Y}}(d) \,. \tag{3.3}$$

Using equations (2.10), (3.2) and (3.3) it is possible to relate the solution at the uppermost surface $x_2 = h + d$ in terms of the solution at the interface $x_2 = d$ and then in terms of the solution at the mid-plane, yielding

$$\mathbf{Y}(d+h) = \mathbf{P}(h)\tilde{\mathbf{P}}(d)\tilde{\mathbf{Y}}(0) \,. \tag{3.4}$$

The dispersion relation for the propagation of flexural waves in the laminate is obtained by satisfying boundary and continuity conditions as well as incorporating the requirements that $\tilde{U}$ and $\tilde{\tau}_2$ vanish at $x_2 = 0$, equation (3.4) then reducing to

$$\left.\begin{pmatrix} -iU \\ V \\ 0 \\ 0 \end{pmatrix}\right|_{x_2 = h+d} = \mathbf{P}(h)\tilde{\mathbf{P}}(d) \left.\begin{pmatrix} 0 \\ \tilde{V} \\ \frac{\tilde{\tau}_1}{ik} \\ 0 \end{pmatrix}\right|_{x_2 = 0} \,. \tag{3.5}$$

This yields a system of four equations, for which a non-trivial solution exists provided

$$P_{3i}\tilde{P}_{i2}P_{4j}\tilde{P}_{j3} - P_{3i}\tilde{P}_{i3}P_{4j}\tilde{P}_{j2} = 0 \,, \tag{3.6}$$

where (3.6) is the dispersion equation for flexural waves in the laminate. On inserting the definitions of P_{ij} (and $\tilde{P}_{ij}$) into (3.6), this relation is expressible in the form

$$\begin{aligned}&2q_1q_2f(q_1)f(q_2)d_1 + q_1f(q_2)^2\{-C_1S_2d_2 + C_1C_2d_3 + S_1S_2d_4 - S_1C_2d_5\}\\&+ q_2f(q_1)^2\{S_1C_2d_2 - S_1S_2d_3 - C_1C_2d_4 + C_1S_2d_5\} = 0\ ,\end{aligned} \tag{3.7}$$

within which

$$\begin{aligned}d_1 &= p_1q_2\{\tilde{f}(p_2) - f(q_2)\}\{f(q_1) - \tilde{f}(p_2)\}\tilde{S}_1\tilde{C}_2 + p_2q_2\{\tilde{f}(p_1) - f(q_2)\}\{\tilde{f}(p_1) - f(q_1)\}\tilde{C}_1\tilde{S}_2\ ,\\
d_2 &= p_1p_2\{f(q_2) - f(q_1)\}\{\tilde{f}(p_1) - \tilde{f}(p_2)\}\tilde{S}_1\tilde{S}_2\ ,\\
d_3 &= p_1q_2\{\tilde{f}(p_2) - f(q_1)\}^2\tilde{S}_1\tilde{C}_2 - p_2q_2\{\tilde{f}(p_1) - f(q_1)\}^2\tilde{C}_1\tilde{S}_2\ ,\\
d_4 &= p_2q_1\{\tilde{f}(p_1) - f(q_2)\}^2\tilde{C}_1\tilde{S}_2 - p_1q_1\{\tilde{f}(p_2) - f(q_2)\}^2\tilde{S}_1\tilde{C}_2,\\
d_5 &= q_1q_2\{f(q_2) - f(q_1)\}\{\tilde{f}(p_1) - \tilde{f}(p_2)\}\tilde{C}_1\tilde{C}_2.\end{aligned}$$

The main motivation for expressing the dispersion relation in the form shown in equation (3.7) is that each of $d_i(i = 1, \ldots, 6)$ are independent of σ_2, which simplifies the subsequent analysis of quasi-static solutions and stability. As we shall see later, this representation also helps considerably in the examination of the various asymptotic limits, especially the short wavelength limit $(kh, kd \to \infty)$.

It is noted that q_1 and q_2 (p_1 and p_2) may be either real, purely imaginary or complex conjugates. The implication is that there exists twenty five distinct cases to consider if one wishes to solve the dispersion equation numerically. However, in each case the dispersion relation remains either real or purely imaginary. This is shown most easily by considering the components of the propagator matrices $\mathbf{P}(h)$ and $\tilde{\mathbf{P}}(d)$. It is easily verified that all components of $\mathbf{P}(h)$ (and similarly $\tilde{\mathbf{P}}(d)$) remain real in the cases when one or both of q_1 and q_2 (p_1 and p_2) are purely imaginary. In the case when q_1 and q_2 (p_1 and p_2) are complex conjugates each component of $\mathbf{P}(h)$ ($\tilde{\mathbf{P}}(d)$) is a quotient formed by the difference of two complex conjugates, thus ensuring that the components of $\mathbf{P}(h)$ ($\tilde{\mathbf{P}}(d)$) all remain real in this case. It is also noted that the removal of a common factor from the dispersion relation (3.7) means that whilst all the components of each propagator matrix are always real, the dispersion relation as expressed in the form of equation (3.7) will either be real or purely imaginary.

4 Analysis of the dispersion relation

4.1 Long wave limit $\mathbf{kh}, \mathbf{kd} \to 0$

The long wave limit of (3.7) is first investigated by allowing $kh, kd \to 0$, whilst assuming that the speed of the wave propagation remains finite, the leading order term then being

$$\left\{p_2^2\tilde{f}(p_1)^2 - p_1^2\tilde{f}(p_2)^2\right\}\{f(q_1) - f(q_2)\}\,d + \{q_2^2f(q_1)^2 - q_1^2f(q_2)^2\}\left\{\tilde{f}(p_1) - \tilde{f}(p_2)\right\}h = 0. \tag{4.1}$$

After a little algebraic manipulation, and use of equation (2.8), the long wavelength limiting wave speed may be shown to be

$$v^2 = \frac{\left\{\tilde{\alpha} - \tilde{\gamma}\left(1 - \frac{\sigma_2}{\tilde{\gamma}}\right)^2\right\}d + \left\{\alpha - \gamma\left(1 - \frac{\sigma_2}{\gamma}\right)^2\right\}h}{\tilde{\rho}d + \rho h}. \tag{4.2}$$

In obtaining equation (4.2) it is noted that four common factors, namely $q_1 = 0$, $q_2 = 0$, $q_1 = q_2$ and $p_1 = p_2$, have been removed, these factors corresponding to spurious non-dispersive roots. Specifically, the first two correspond to exceptional shear waves, whilst the latter two afford only a trivial solution of the boundary conditions. As may be expected the single plate result found by Rogerson and Fu [19] may be recovered by setting $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$ and $\rho = \tilde{\rho}$ in equation (4.2).

Numerical results indicate that whilst the fundamental mode tends to the finite limit indicated by (4.2) as $kh, kd \to 0$, all harmonics have an associated infinite wave speed limit. This has been shown analytically to be the case for a single plate, see Rogerson and Fu [19], in which asymptotic expansions for the wave speeds associated with each harmonic were derived for the low wave number regime. In our present problem the increased algebraic complexity makes such expansions difficult and time consuming to obtain, however it has been possible to verify numerically that for all harmonics the phase speed tends to infinity as $kh, kd \to 0$.

4.2 Shortwave limit $\mathbf{kh, kd} \to \infty$

Consider first the limit $kh, kd \to \infty$ in the case when q_1 and q_2 (p_1 and p_2) are either real or complex conjugates. A relatively trivial examination of equation (2.7) shows that q_1 and q_2 may take complex values only if ρv^2 lies within in the region $\lambda^- \leq \rho v^2 \leq \lambda^+ \leq \alpha$, where λ^- and λ^+ are roots of the quadratic equation

$$\lambda^2 + 4\lambda(\gamma - \beta) + 4(\beta^2 - \alpha\gamma) = 0, \tag{4.3}$$

and it is also noted that if $\alpha > 2\beta$ then $2\beta < \lambda^+ \leq \alpha$. In the limit $kh, kd \to \infty$ the dispersion relation (3.7) tends to

$$\left\{q_1 f(q_2)^2 - q_2 f(q_1)^2\right\}\left\{d_2^\infty - d_3^\infty - d_4^\infty + d_5^\infty\right\} = 0, \tag{4.4}$$

in which the superscript ∞ indicates that the hyperbolic functions in equation (3.7) are now replaced by unity.

After a little algebraic manipulation equation (4.4) reduces to either $R(v) = 0$ or $S(v) = 0$ where

$$R(v) = q_1\left\{\gamma(q_2^2+1) - \sigma_2\right\}^2 - q_2\left\{\gamma(q_1^2+1) - \sigma_2\right\}^2, \tag{4.5}$$

$$\begin{aligned} S(v) = {} & \gamma^2\left\{q_1q_2(q_1+q_2)^2 - (q_1q_2-1)^2\right\} + \tilde{\gamma}^2\left\{p_1p_2(p_1+p_2)^2 - (p_1p_2-1)^2\right\} \\ & + \gamma\tilde{\gamma}\left\{2(p_1p_2-1)(q_1q_2-1) + (p_1p_2+q_1q_2)(q_1+q_2)(p_1+p_2)\right\}. \end{aligned} \tag{4.6}$$

Equation (4.5) is the Rayleigh surface wave speed equation for the upper layer derived by Dowaikh and Ogden [7], whilst equation (4.6) is the Stoneley interfacial wave speed equation derived by Dowaikh and Ogden [10]. Whether such waves exist is dependent on both material parameters and the value of σ_2. However, if both waves exist the fundamental mode and the first harmonic will each tend to one of the corresponding wave speeds, with the fundamental tending to the lower. If, on the other hand, only one of these waves exist the corresponding wave speed is the limit of the fundamental mode.

Numerical results indicate that the short wavelength limiting wave speed for all harmonics is in general the lower of two shear wave speeds, termed limiting wave speeds, associated with the inner core and the outer layers, respectively. It is reiterated that an exception arises when both a surface and interfacial wave exist, in which case all harmonics higher than the first tend to the limit indicated above. The value of these limiting wave speeds within each layer is dependent on the relative magnitudes of α and 2β ($\tilde{\alpha}$ and $2\tilde{\beta}$). Initial numerical calculations indicated that as $kh, kd \to \infty$ one of q_1, q_2 was imaginary, the other real, and if $q_1 = i\hat{q}_1$, where $\hat{q}_1 \geq 0$ is real, then $\hat{q}_1 \to 0$ with $q_2 \to (2\beta - \alpha)/\gamma$. This scenario is exemplified by earlier studies of materials for which $\alpha + \gamma = 2\beta$ and $\tilde{\alpha} + \tilde{\gamma} = 2\tilde{\beta}$, see e.g. Rogerson and Sandiford [16]. However, this is only valid provided $2\beta \geq \alpha$. In the case $2\beta < \alpha$ it may be shown that both q_1 and q_2 are purely imaginary as $kh, kd \to \infty$, and furthermore as $kh \to \infty$, $|q_1| \to |q_2|$. The limiting wave speed in the outer layer may therefore be written explicitly as

$$\rho v_L^2 = \begin{cases} \alpha & \alpha \le 2\beta \\ 2\beta - 2\gamma + 2\sqrt{\gamma}\sqrt{\alpha + \gamma - 2\beta} & \alpha > 2\beta \end{cases}, \tag{4.7}$$

with the corresponding limiting wave speed for the inner core given by

$$\tilde{\rho}\tilde{v}_L^2 = \begin{cases} \tilde{\alpha} & \tilde{\alpha} \le 2\tilde{\beta} \\ 2\tilde{\beta} - 2\tilde{\gamma} + 2\sqrt{\tilde{\gamma}}\sqrt{\tilde{\alpha} + \tilde{\gamma} - 2\tilde{\beta}} & \tilde{\alpha} > 2\tilde{\beta} \end{cases}. \tag{4.8}$$

It is worth noting that the limiting wave speed shown in equation (4.7) agrees with that obtained by Ogden and Sotiropoulos [11]. These authors investigated the related problem of a plate bonded to a half space and imposed the decay conditions $q_1^2 q_2^2 \ge 0$ and $(q_1 + q_2)^2 \ge 0$ and found (4.7) to be the upper bound of the speed of interfacial waves. It is clear that in general there are four possible wave speed limits for the harmonics as $kh, kd \to \infty$ and the actual limit is dependent on material parameters. We shall now analyse each of the four possible limits in turn.

4.2.1 Case 1: $2\tilde{\beta} \ge \tilde{\alpha}$ and $\tilde{\rho}\tilde{\mathbf{v}}_L^2 < \rho \mathbf{v}_L^2$

In the case $2\tilde{\beta} \ge \tilde{\alpha}$ and $\tilde{\rho}\tilde{v}_L^2 < \rho v_L^2$ numerical calculations indicate that for all harmonics $\tilde{\rho}v^2 \to \tilde{\alpha}$ from above and therefore only one of p_1 and p_2 is real, the other being purely imaginary, with q_1 and q_2 either real or complex conjugates. Accordingly, and without loss of generality, it is assumed that $p_1 = i\hat{p}_1$, where $\hat{p}_1 \ge 0$ is real and $\hat{p}_1 \to 0$ as $kh, kd \to \infty$. Equation (3.7) may now be invoked to deduce that as $kh, kd \to \infty$ either $R(v) = 0$ or

$$\hat{p}_1 \tan(k\hat{p}_1 d) \left\{\eta^{(1)}(p_2, q_1, q_2) + O(\hat{p}_1)\right\} = \zeta^{(1)}(p_2, q_1, q_2) + O(\hat{p}_1), \tag{4.9}$$

with

$$\begin{aligned} \eta^{(1)}(p_2, q_1, q_2) =& q_2 \left\{\gamma(q_1^2 + 1) - \tilde{\gamma}(p_2^2 + 1)\right\}^2 - q_1 \left\{\gamma(q_2^2 + 1) - \tilde{\gamma}(p_2^2 + 1)\right\}^2 \\ &+ p_2^3(q_2^2 - q_1^2)\gamma\tilde{\gamma}, \\ \zeta^{(1)}(p_2, q_1, q_2) =& p_2^2 q_1 q_2 (q_2^2 - q_1^2)\gamma\tilde{\gamma} - p_2 q_2 \left\{\gamma(q_1^2 + 1) - \tilde{\gamma}\right\}^2 \\ &+ p_2 q_1 \left\{\gamma(q_2^2 + 1) - \tilde{\gamma}\right\}^2. \end{aligned}$$

It is noted that the Rayleigh wave speed is not a valid solution as $\tan(k\hat{p}_1 d)$ cannot be bounded as $kd \to \infty$ if $v_R > \tilde{v}_S$ and v_R is the asymptotic limit. Furthermore, if $v_R < \tilde{v}_S$, and v_R were the asymptotic limit, then the assumption that p_1 is imaginary would be violated. However, in such cases when a Rayleigh surface wave speed exists a surface wave front may be formed from the cumulative contribution of harmonics. This interesting point is discussed further in the numerical section.

From equation (4.9) it is readily deduced that $O(\hat{p}_1)\tan(k\hat{p}_1 d) \sim O(1)$, implying that $\tan(k\hat{p}_1 d) \to \infty$ as $\hat{p}_1 \to 0$, and therefore

$$\hat{p}_1 = \frac{\left(n + \frac{1}{2}\right)\pi}{kd} + O(kd)^{-2}. \tag{4.10}$$

In order to obtain an expansion for the phase speed use must be made of the appropriate form of equation (2.7), which may be written as

$$\tilde{\rho}v^2 = \frac{\tilde{\gamma}\hat{p}_1^4 + 2\tilde{\beta}\hat{p}_1^2 + \tilde{\alpha}}{1 + \hat{p}_1^2}. \tag{4.11}$$

On inserting equation (4.10) into (4.11) an approximation to the phase speed is obtained, namely

$$\tilde{\rho}v^2 \approx \tilde{\alpha} + \left(2\tilde{\beta} - \tilde{\alpha}\right)\left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{(kd)^2}. \tag{4.12}$$

A higher order approximation may be obtained by setting

$$\hat{p}_1 = \left(n + \frac{1}{2}\right)\frac{\pi}{kd} + \frac{\phi_1}{(kd)^2} + O(kd)^{-3}\ , \quad \tan\left(k\hat{p}_1 d\right) = -\frac{kd}{\phi_1} + O(kd)^{-1}, \tag{4.13}$$

in which ϕ_1 is yet to be determined. On inserting these two expansions into (4.9), and equating leading order terms, it is deduced that

$$\phi_1 = -\frac{\eta^{(1)}(\bar{p}_2, \bar{q}_1, \bar{q}_2)}{\zeta^{(1)}(\bar{p}_2, \bar{q}_1, \bar{q}_2)}\left(n + \frac{1}{2}\right)\pi, \tag{4.14}$$

where $\bar{p}_2$, $\bar{q}_1$ and $\bar{q}_2$ are defined as

$$\bar{p}_2^2 = \frac{2\tilde{\beta} - \tilde{\alpha}}{\tilde{\gamma}}\ , \quad \bar{q}_1, \bar{q}_2 = \frac{(2\beta - \bar{\rho}\tilde{\alpha}) \pm \sqrt{(2\beta - \bar{\rho}\tilde{\alpha})^2 - 4\gamma(\alpha - \bar{\rho}\alpha)}}{2\gamma}\ , \tag{4.15}$$

with $\bar{\rho} = \frac{\rho}{\tilde{\rho}}$. It is noted that the order of the correction term would have to be modified in the case when $2\beta \approx \alpha$. However, further discussion of this point will be deferred until some other occasion. Equation (4.14) may now be invoked, in conjunction with equations (4.11) and $(4.13)_1$, to obtain

$$\tilde{\rho}v^2 \approx \tilde{\alpha} + \left(2\tilde{\beta} - \tilde{\alpha}\right)\left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{(kd)^2}\left\{1 - \left(\frac{2}{kd}\right)\frac{\eta^{(1)}(\bar{p}_2, \bar{q}_1, \bar{q}_2)}{\zeta^{(1)}(\bar{p}_2, \bar{q}_1, \bar{q}_2)}\right\}. \tag{4.16}$$

4.2.2 Case 2: $2\beta \geq \alpha$ and $\rho \mathbf{v}_L^2 < \tilde{\rho}\, \tilde{\mathbf{v}}_L^2$

By an argument similar to the previous case, only one of q_1 and q_2 is now imaginary, whilst p_1 and p_2 are either real or complex conjugates as $kh, kd \to \infty$. From equation (3.7) it is deduced that if $q_1 = i\hat{q}_1$, with $\hat{q} \geq 0$, that as $kd, kd \to \infty$

$$\begin{aligned} &\tan\left(k\hat{q}_1\right)\left\{(\gamma - \sigma_2)^2 q_2 \eta^{(2)}(p_1, p_2, q_2) + O(\hat{q}_1)\right\} = \\ &\qquad \hat{q}_1\left\{\left(\gamma(q_2^2 + 1) - \sigma_2\right)^2 \eta^{(2)}(p_1, p_2, q_2) + (\gamma - \sigma_2)^2 \zeta^{(2)}(p_1, p_2, q_2) + O(\hat{q}_1)\right\}\ , \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} \eta^{(2)}(p_1, p_2, q_2) =& p_1 q_2\{\gamma - \tilde{\gamma}(p_2^2 + 1)\}^2 - p_2 q_2\{\gamma - \tilde{\gamma}(p_1^2 + 1)\}^2 \\ &+ p_1 p_2 q_2 (p_2^2 - p_1^2)\gamma\tilde{\gamma}\ , \\ \zeta^{(2)}(p_1, p_2, q_2) =& p_1 q_2\{\gamma(q_2^2 + 1) - \tilde{\gamma}(p_2^2 + 1)\}^2 - p_2 q_2\{\gamma(q_2^2 + 1) - \tilde{\gamma}(p_1^2 + 1)\}^2 \\ &- q_2^4 (p_2^2 - p_1^2)\gamma\tilde{\gamma}\ . \end{aligned}$$

From equation (4.17) it is deduced that $O(1)\tan\left(k\hat{q}_1 h\right) \sim O(\hat{q}_1)$, implying that $\tan\left(k\hat{q}_1 h\right) \to 0$ as $\hat{q}_1 \to 0$, and therefore

$$\hat{q}_1 = \frac{n\pi}{kh} + O(kh)^{-2}. \tag{4.18}$$

Inserting equation (4.18) into the appropriate form of equation (4.11) yields the approximation

$$\rho v^2 \approx \alpha + (2\beta - \alpha)\left(\frac{n\pi}{kh}\right)^2\ , \tag{4.19}$$

which may be improved by setting

$$\hat{q}_1 = \left(\frac{n\pi}{kh}\right) + \frac{\phi_2}{(kh)^2} + O(kh)^{-3}\,, \qquad \tan\left(k\hat{q}_1 h\right) = \frac{\phi_2}{kh} + O(kh)^{-3}\,, \tag{4.20}$$

where ϕ_2 is yet to be determined. If these two expansions are inserted into equation (3.7) and like powers of kh equated it is found that

$$\phi_2 = \left\{ \frac{\{\gamma(\bar{q}_2^2+1)-\sigma_2\}^2}{\bar{q}_2(\gamma-\sigma_2)^2} + \frac{\zeta^{(2)}(\bar{p}_1,\bar{p}_2,\bar{q}_2)}{\bar{q}_2\eta^{(2)}(\bar{p}_1,\bar{p}_2,\bar{q}_2)} \right\} n\pi\,, \tag{4.21}$$

where

$$\bar{q}_2^2 = \frac{2\beta-\alpha}{\gamma}\,, \quad \bar{p}_1^2, \bar{p}_2^2 = \frac{2\tilde{\beta}-\bar{\rho}^{-1} \pm \sqrt{(2\tilde{\beta}-\bar{\rho}^{-1})^2 - 4\gamma(\tilde{\alpha}-\bar{\rho}^{-1}\alpha)}}{2\tilde{\gamma}}\,.$$

On inserting equation (4.21) into equation $(4.20)_1$, and making use of the appropriate form of equation (4.11), it may be shown that

$$\rho v^2 \approx \alpha + (2\beta-\alpha)\left(\frac{n\pi}{kh}\right)^2 \left\{1 + \frac{2}{kh}\left\{ \frac{\{\gamma(\bar{q}_2^2+1)-\sigma_2\}^2}{\bar{q}_2(\gamma-\sigma_2)^2} + \frac{\zeta^{(2)}(\bar{p}_1,\bar{p}_2,\bar{q}_2)}{\bar{q}_2\eta^{(2)}(\bar{p}_1,\bar{p}_2,\bar{q}_2)} \right\}\right\}\,. \tag{4.22}$$

4.2.3 Case 3: $\tilde{\alpha} > 2\tilde{\beta}$ and $\tilde{\rho}\tilde{v}_L^2 < \rho v_L^2$

In the case in which both $\tilde{\alpha} > 2\tilde{\beta}$ and $\tilde{\rho}\tilde{v}_L^2 < \rho v_L^2$ our numerical calculations guide us in assuming that as $kh, kd \to \infty$ both p_1 and p_2 are imaginary, $|p_1| \to |p_2|$, whilst q_1 and q_2 are either real or complex conjugates. This limit may therefore be examined by setting

$$p_1^2 = -\tilde{a} + \tilde{b}, \quad p_2^2 = -\tilde{a} - \tilde{b}, \quad \tilde{a} > 0,\ \tilde{b} \ge 0, \tag{4.23}$$

where $\tilde{a}$ and $\tilde{b}$ are real and $\tilde{b} \to 0$ as $kh, kd \to \infty$. The implication is that $\tilde{\rho}v^2 \to \tilde{\lambda}^+$ from above, see equation (4.3), and we note that the region in which the phase speed must lie for equation (4.23) to be valid is $\tilde{\lambda}^+ < \tilde{\rho}v^2 < \tilde{\alpha}$. The values of $\tilde{a}$ and $\tilde{b}$ may be obtained explicitly from the appropriate form of equation (2.7), thus

$$\tilde{a} = \frac{\tilde{\rho}v^2 - 2\tilde{\beta}}{2\tilde{\gamma}}, \quad \tilde{b} = \frac{\sqrt{(2\tilde{\beta}-\tilde{\rho}v^2)^2 - 4\tilde{\gamma}(\tilde{\alpha}-\tilde{\rho}v^2)}}{2\tilde{\gamma}}. \tag{4.24}$$

If the forms of p_1^2 and p_2^2 shown in equation (4.23) are inserted into the appropriate form of equation (2.8) it is observed that

$$\tilde{a} = \tilde{a}_0 + \tilde{a}_1\tilde{b}^2 + O(\tilde{b}^4), \tag{4.25}$$

where

$$\tilde{a}_0 = -1 + \tilde{\gamma}^{-\frac{1}{2}}\left(\tilde{\alpha}+\tilde{\gamma}-2\tilde{\beta}\right)^{\frac{1}{2}}, \quad \tilde{a}_1 = \frac{1}{2(\tilde{a}_0+1)}.$$

On inserting equations (4.23) and (4.25) into equation (3.7) and it is deduced that for high wave number either $R(v) = 0$ or

$$\begin{aligned} 2\gamma\tilde{\gamma}\left((q_2^{(0)})^2 - (q_1^{(0)})^2\right)(q_1^{(0)}q_2^{(0)} - \tilde{a}_0)\tilde{b}C(\tilde{a}_0) + \tilde{b}S(\tilde{a}_0)\left\{2\sqrt{\tilde{a}_0}\chi^{(1)} - \frac{\chi^{(0)}}{\sqrt{\tilde{a}_0}}\right\} \\ = 2\sqrt{\tilde{a}_0}\chi^{(0)}S(\tilde{b}) - 2\gamma\tilde{\gamma}\left((q_2^{(0)})^2 - (q_1^{(0)})^2\right)\left(\tilde{a}_0 + q_1^{(0)}q_2^{(0)}\right)\tilde{b}C(\tilde{b}) + O(\tilde{b}^2), \end{aligned} \tag{4.26}$$

where

$$
\begin{aligned}
\chi^{(0)} &= q_2^{(0)}\left\{\gamma\left((q_1^{(0)})^2+1\right)+\tilde{\gamma}(\tilde{a}_0-1)\right\}^2 - q_1^{(0)}\left\{\gamma((q_2^{(0)})^2+1\right)+\tilde{\gamma}(\tilde{a}_0-1)\right\}^2,\\
\chi^{(1)} &= 2\tilde{\gamma}\left(q_2^{(0)}\left\{\gamma\left((q_1^{(0)})^2+1\right)+\tilde{\gamma}(\tilde{a}_0-1)\right\} - q_1^{(0)}\left\{\gamma\left((q_2^{(0)})^2+1\right)+\tilde{\gamma}(\tilde{a}_0-1)\right\}\right),\\
\chi^{(2)} &= \tilde{a}_1\chi^{(1)}+\tilde{\gamma}^2(q_2^{(0)}-q_1^{(0)}),
\end{aligned}
$$

$$
C(\tilde{a}_0)=\cos\left(2\sqrt{\tilde{a}_0}kd\right),\ \ S(\tilde{a}_0)=\sin\left(2\sqrt{\tilde{a}_0}kd\right),\ \ C(\tilde{b})=\cos\left(\frac{\tilde{b}kd}{\sqrt{\tilde{a}_0}}\right),\ \ S(\tilde{b})=\sin\left(\frac{\tilde{b}kd}{\sqrt{\tilde{a}_0}}\right),
$$

and $q_1^{(0)}, q_2^{(0)}$ are in general $O(1)$ terms which may be found by setting $\rho v^2 = \tilde{\lambda}^+/\tilde{\rho}$ in equation (2.7). The Rayleigh surface wave is rejected as a solution for a similar argument to that given in a preceding section. From equation (4.26) it is readily deduced that $O(\tilde{b}) \sim S(\tilde{b})$, implying that $S(\tilde{b}) \to 0$ as $\tilde{b} \to 0$, and therefore

$$
\tilde{b} = \frac{\sqrt{\tilde{a}_0}n\pi}{kd} + O(kd)^{-2}. \tag{4.27}
$$

A second order approximation for the phase speed may now be obtained by inserting equation (4.27) into equations $(4.24)_1$ and (4.25), to obtain

$$
\tilde{\rho}v^2 \approx 2\tilde{\beta} - 2\tilde{\gamma} + 2\sqrt{\tilde{\gamma}}\sqrt{\tilde{\alpha}+\tilde{\gamma}-2\tilde{\beta}} + \left(\frac{n\pi}{kd}\right)^2\left\{\frac{\tilde{\gamma}\tilde{a}_0}{\tilde{a}_0+1}\right\}. \tag{4.28}
$$

A higher order approximation may be found by setting

$$
\tilde{b} = \sqrt{\tilde{a}_0}\frac{n\pi}{kd} + \frac{\phi_3}{(kd)^2} + O(kd)^{-3} \tag{4.29}
$$

from which we infer that

$$
\sin\left(\frac{\tilde{b}kd}{\sqrt{\tilde{a}_0}}\right) = (-1)^n\frac{\phi_3}{\sqrt{\tilde{a}_0}kd} + O(kd)^{-3}, \quad \cos\left(\frac{\tilde{b}kd}{\sqrt{\tilde{a}_0}}\right) = (-1)^n + O(kd)^{-2}, \tag{4.30}
$$

where ϕ_3 is to be determined. On inserting equation (4.29) into equation (4.26) and comparing like powers of kd it is established that

$$
\begin{aligned}
\phi_3 = (-1)^n\frac{\sqrt{\tilde{a}_0}}{2\chi^{(0)}}\Bigg\{&2\gamma\tilde{\gamma}\left((q_2^{(0)})^2-(q_1^{(0)})^2\right)\left\{(-1)^n\left(\tilde{a}_0+q_1^{(0)}q_2^{(0)}\right)+\left(q_1^{(0)}q_2^{(0)}-\tilde{a}_0\right)C(\tilde{a}_0)\right\}\\
&+\sqrt{\tilde{a}_0}\left(2\chi^{(1)}-\frac{\chi^{(0)}}{\tilde{a}_0}\right)S(\tilde{a}_0)\Bigg\}n\pi.
\end{aligned} \tag{4.31}
$$

Equation (4.31) may now be used, in conjunction with equations $(4.24)_1$, (4.25) and (4.29), to obtain

$$
\tilde{\rho}v^2 \approx 2\tilde{\beta} - 2\tilde{\gamma} + 2\sqrt{\tilde{\gamma}}\sqrt{\tilde{\alpha}+\tilde{\gamma}-2\tilde{\beta}} + \left(\frac{n\pi}{kd}\right)^2\left\{\frac{\tilde{\gamma}}{\tilde{a}_0+1}\right\}\left\{\tilde{a}_0+\frac{2\sqrt{\tilde{a}_0}\hat{\phi}_3}{kd}\right\}, \tag{4.32}
$$

where $\hat{\phi}_3 = \phi_3/n\pi$.

4.2.4 Case 4: $\alpha > 2\beta$ and $\rho \mathbf{v}_{\mathbf{L}}^{2} < \tilde{\rho}\tilde{\mathbf{v}}_{\mathbf{L}}^{2}$

In the case in which $\alpha > 2\beta$ and $\rho v_L^2 \le \tilde{\rho}\tilde{v}_L^2$ both q_1 and q_2 are imaginary, with p_1 and p_2 either real or complex conjugates in the short wavelength limit. In addition, numerical results indicate that as $kh, kd \to \infty$, $|q_1| \to |q_2|$ and as such the analogous form of equations (4.23)–(4.25) may be used to show that as $kh, kd \to \infty$ equation (3.7) takes the form

$$\begin{aligned} &4\hat{q}_1\hat{q}_2 f(q_1)f(q_2)d_1 + \\ &\left\{\hat{q}_1 f(q_2)^2 - \hat{q}_2 f(q_1)^2\right\}\left\{(d_5 - d_2)\sin\{2(\sqrt{a_0} + \xi b^2)kh\} + (d_3 + d_4)\cos\{2(\sqrt{a_0} + \xi b^2)kh\}\right\} \\ &\qquad = \left\{\hat{q}_1 f(q_2)^2 + \hat{q}_2 f(q_1)^2\right\}\left\{(d_2 + d_5)S(b) + (d_4 - d_3)C(b)\right\} + O(b^3), \end{aligned} \tag{4.33}$$

where $\xi = (4a_0a_1 - \phi)/a_0^{\frac{3}{2}}$ and $S(b)$ and $C(b)$ may be inferred from the definition given directly after equation (4.26). After some algebraic manipulation, and making use of the appropriate form of equations $(4.24)_1$ and (4.25), equation (4.33) may be written as

$$A_1 + (A_2 - A_3C(a_0) - A_4S(a_0))\, b^2 = (A_1 + A_5b^2)C(b) + A_6bS(b) + O(b^3), \tag{4.34}$$

in which A_m are order one quantities given explicitly in the appendix.

To leading order it is deduced from equation (4.34) that $\left(1 - \cos\left(\frac{bkh}{\sqrt{a_0}}\right)\right) \sim O(bS(b))$, thus implying that $\cos\left(\frac{bkh}{\sqrt{a_0}}\right) \to 1$ as $kh, kd \to \infty$, and therefore

$$b = 2\sqrt{a_0}\frac{n\pi}{kh} + O(kh)^{-2}. \tag{4.35}$$

A second order approximation to the phase speed may be found by inserting equation (4.35) into the appropriate form of equation $(4.24)_1$, after making use of the appropriate form of equation (4.25), to yield

$$\rho v^2 \approx 2\beta - 2\gamma + 2\sqrt{\gamma}\sqrt{\alpha + \gamma - 2\beta} + \left\{\frac{\gamma a_0}{a_0 + 1}\right\}\left(\frac{n\pi}{kh}\right)^2. \tag{4.36}$$

A higher order expansion may be found by setting

$$b = 2\sqrt{a_0}\frac{n\pi}{kh} + \frac{\phi_4}{(kh)^2} + O(kh)^{-3}, \tag{4.37}$$

from which it is deduced

$$\sin\left(\frac{bkh}{\sqrt{a_0}}\right) = \frac{\phi_4}{\sqrt{a_0}kh} + O(kh)^{-3}, \quad \cos\left(\frac{bkh}{\sqrt{a_0}}\right) = 1 - \frac{\phi_4^2}{2a_0(kh)^2} + O(kh)^{-4}, \tag{4.38}$$

where ϕ_4 is to be determined and we note it is now necessary to have an $O(kh)^{-2}$ term in the expansion $(4.38)_2$. Inserting equation (4.35) and (4.37) into equation (4.34) and comparing like powers of kh reveals that the equation is identically zero at leading order, the next order yielding the following quadratic equation for ϕ_4

$$\frac{A_1}{2a_0}\phi_4^2 - 2A_6n\pi\phi_4 + 4a_0n^2\pi^2\left\{A_2 - A_5 - A_3C(a_0) - A_4S(a_0)\right\} = 0. \tag{4.39}$$

It is interesting to note that in this case a quadratic equation for ϕ_4 is obtained, whilst in the previous three cases a single value for ϕ_1, ϕ_2 and ϕ_3 was obtained. However, it has been verified numerically that for particular values of n the two solutions indicated in equation (4.39) correspond to two distinct branches of the dispersion relation. Indeed the second order approximation to the phase speed in equation (4.36) gives asymptotic solutions which pass between pairs of harmonics, thus in order to obtain accurate asymptotic solutions the expansion *must* be taken to third order, whilst in the previous three cases a reasonable approximation may be found from the second order expansion. Equation (4.39) may therefore be used, in conjunction with the appropriate form of equations $(4.24)_1$ and (4.25), to obtain

$$\rho v^2 = \begin{cases} 2\beta - 2\gamma + 2\sqrt{\gamma}\sqrt{\alpha+\gamma-2\beta} + \dfrac{(n+1)^2\pi^2}{4(kh)^2}\left\{\dfrac{4\gamma}{a_0+1}\right\}\left\{a_0 + \dfrac{\sqrt{a_0}\hat{\phi}_4^-}{kh}\right\} & \text{n odd} \\ 2\beta - 2\gamma + 2\sqrt{\gamma}\sqrt{\alpha+\gamma-2\beta} + \dfrac{n^2\pi^2}{4(kh)^2}\left\{\dfrac{4\gamma}{a_0+1}\right\}\left\{a_0 + \dfrac{\sqrt{a_0}\hat{\phi}_4^+}{kh}\right\} & \text{n even} \end{cases}, \tag{4.40}$$

where $\hat{\phi}_4 = \phi_4/n\pi$, ϕ_4^+ and ϕ_4^- representing solutions of equation (4.39) associated with the positive and negative square root, respectively.

5 Some numerical results

Throughout this section numerical results are presented which illustrate the character of the dispersion relation discussed in the previous section. For the material parameters employed in figure 1 an example is afforded of a situation in which a real value of v_R exists such that $v_R > \tilde{v}_L$. In this case the short wave length $(kh, kd \to \infty)$ limiting wave speed of all branches is $\tilde{v}_L$. It is noted that as each harmonic crosses from above to below $v_L = 1.732$, the nature of the disturbance in the outer layer changes from sinusoidal variation to exponential variation with depth. In contrast the nature of the disturbance in the core is always sinusoidal.

Figures 1(a) and 1(b) about here

A striking feature of figure 1(a) is the distinct plateau below $v = v_L$ which are formed by the flattening of the various harmonics. These harmonics combine to produce what at first glance looks like a straight line at the constant Rayleigh surface wave speed associated with the outer layer. This phenomenon is further elucidated by examining the associated group velocity curves in figure 1(b). These clearly show that each harmonic higher than the second exhibits an extremely flat maximum at the surface wave speed. Furthermore, these flat maxima are within adjacent wave number regions and the cumulative effect of the harmonics will give rise to a wave front travelling with speed v_R, with associated displacements and stresses which decay with depth. A further point of note is that the higher harmonics are beginning to form a wave front associated with the shear wave in the outer layer.

Figures 2(a), 2(b) and 2(c) about here

Some comparisons of numerical and asymptotic solutions will now be discussed. Specifically, attention is focused on cases 3 and 4, these two cases exhibiting behaviour radically different from earlier studies concerning restricted classes of incompressible material. For the specific material parameters used in figure 2 the material response is that of case 3. The high wave number limiting wave speed of all branches in this case is the limiting speed associated with the core. In figure 2(a) the sinusoidal behaviour noted to occur at third order in the asymptotic expansion (4.32) is clearly evident in the moderate wave number regime. Some flattening of the harmonics in the region of $v = 2.0$ and $v = \sqrt{3}$ is observed, these being the speed of shear waves associated with $q = 0$ and $p = 0$, respectively. Within figure 2(b) the second order expansions (4.32) are shown for the first four harmonics and are clearly not capable of fully representing the sinusoidal behaviour. When the improved third order expansions are plotted in figure 3(c) the agreement with the numerical solution is exceptionally good, with in particular the oscillatory behaviour clearly manifested in the asymptotic solution.

Figures 3(a), 3(b) and 3(c) about here

The final set of chosen material parameters yield an example of the type of material response labelled case 4 in the previous section. In figure 3(a) it is seen that the harmonics form distinct pairs in the moderate and high wave number regimes and that sinusoidal behaviour occurs in such a way that each pair move close together and then apart seemingly in phase. It is then the case that the appropriate second order expansion (4.36) yields an approximation to both the $2n^{th}$ and $(2n-1)^{th}$ harmonics. This may be observed in figure 3(b) in which the first four harmonics are shown with the corresponding two approximations obtained from (4.36). The better asymptotic expansions (4.40) are shown in figure 3(c) and are clearly more capable of fully reproducing the observed sinusoidal behaviour.

6 Quasi-static solutions, neutral curves and stability

A significant difference between problems involving waves in pre-stressed and linear elastic laminates is that the former may support standing waves. There therefore exists a bifurcation criterion which defines the states of stress for which quasi-static modes of the primary deformation may exist. This criterion in the present problem is obtained by setting the phase speed equal to zero in the dispersion relation (3.7), thus yielding the quadratic equation

$$\left\{\frac{2d_1}{C_1C_2}+q_1\nu_1+q_2\nu_2\right\}\sigma_2^2-2\left\{(2\beta+2\gamma)\frac{d_1}{C_1C_2}+\gamma q_1(q_2+1)\nu_1+\gamma q_2(q_1^2+1)\nu_2\right\}\sigma_2$$
$$+2(\alpha^2+2\beta\gamma+\gamma^2)\frac{d_2}{C_1C_2}+\gamma^2q_1(q_2^2+1)^2\nu_1+\gamma^2q_2(q_1^2+1)\nu_2=0, \tag{6.1}$$

where

$$\nu_1=-T_2d_2+d_3+T_1T_2d_4-T_1d_5,$$
$$\nu_2=T_1d_2-T_1T_2d_3-d_4+T_2d_5,$$

and it is noted that p_m and q_m now take the values

$$q_1^2,q_2^2=\frac{\beta\pm\sqrt{\beta^2-\alpha\gamma}}{\gamma},\quad p_1^2,p_2^2=\frac{\tilde{\beta}\pm\sqrt{\tilde{\beta}^2-\tilde{\alpha}.\tilde{\gamma}}}{\tilde{\gamma}} \tag{6.2}$$

The solutions of equation (6.1) define two curves in the (σ_2,kh) plane which are usually termed neutral curves. Numerical discussion of the solution of (6.1) will be deferred until later in this section when a specific strain energy function and a simple primary deformation will be considered. However, before we consider this illustrative example the high and low wave number limits of (6.1) will briefly be discussed.

The limit $kh,kd\to\infty$ may be obtained using either equations (6.1) or the appropriate form of the Rayleigh surface wave speed equation (4.5). In either case this leads to two values of σ_2 given by

$$\sigma_2^\infty=\gamma\left\{1-\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{2}}\right\}\pm\sqrt{2\gamma}\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{4}}(\beta+\sqrt{\alpha\gamma})^{\frac{1}{2}}. \tag{6.3}$$

It is noted that in the case when the outer layer is unstrained, i.e. $\alpha=\beta=\gamma=\mu$ (a shear modulus), equation (6.3) reduces to $\sigma_2^\infty=\pm2\mu$, these two values being those between which Dowaikh and Ogden [7] found a unique surface wave in an incompressible elastic half space without a primary deformation.

Consider now the limit $kh,kd\to0$ which may be obtained by formally taking the limit of equation (6.1) or by employing the limiting wave speed equation (4.2), thus in both cases leading to the quadratic equation

$$\left(\sigma_2^0\right)^2(\gamma\xi+\tilde{\gamma})-2\gamma\tilde{\gamma}(1+\xi)\sigma_2^0-\gamma\tilde{\gamma}\left\{(\tilde{\alpha}-\tilde{\gamma})\xi+(\alpha-\gamma)\right\}=0, \tag{6.4}$$

where $\xi = \frac{d}{h}$. It is readily deduced that from equations (6.4) that two real values of σ_2^0 exist if and only if

$$\tilde{\alpha}\gamma\xi^2 + \left(\tilde{\alpha}\tilde{\gamma} + \alpha\gamma - (\gamma - \tilde{\gamma})^2\right)\xi + \alpha\tilde{\gamma} \geq 0\ , \tag{6.5}$$

where it is noted that strong ellipticity requires that α, $\tilde{\alpha}$, γ and $\tilde{\gamma}$ are all positive. It may be further deduced from (6.5) that a simple sufficient condition for the existence of two real values of σ_2^0 is given by $\gamma = \tilde{\gamma}$. Violation of the inequality (6.5) indicates the possible existence of a region within the low wave number regime in which the phase speed will always remain real and bifurcation from the homogeneous deformation is therefore not possible. It is also noted that when real values of σ_2^0 do exist they are both finite. This contrasts with the similar case for extensional waves in which σ_2 may not become complex, however, only one value of σ_2 remains finite in this limit. In addition, it is also noted that the existence of complex σ_2^0 is not a feature of flexural waves in a single plate, see Ogden and Roxburgh [18] and Rogerson and Fu [19].

6.1 A neo-Hookean strain energy function

In this section quasi-static solutions are discussed in the case when each layer is composed of a neo-Hookean material, with the corresponding constitutive part of the strain energy function being therefore of the general form

$$W = \frac{1}{2}C\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right)\ , \tag{6.6}$$

in which C is the shear modulus and λ_i is the principal stretch in the Ox_i direction. In order to further simplify the neutral curves it is assumed that in all layers the principal stretch along Ox_3 is equal to unity, i.e. $\lambda_3 = 1$, $\tilde{\lambda}_3 = 1$ and furthermore that the materials of the outer layer and inner core are identical in their undistorted state and that stretches along Ox_1 and Ox_2 are reversed, therefore

$$\lambda_1 = \tilde{\lambda}_2 = \lambda \ \text{ and } \ \lambda_2 = \tilde{\lambda}_1 = \lambda^{-1}\ . \tag{6.7}$$

With the specialisations indicated by equations (6.6) and (6.7), and those which follow, the two neutral values of σ_2 are deduced from (6.1) to satisfy the quadratic equation

$$\begin{aligned}\left\{\lambda^2\chi_2 + \chi_1 - \frac{4\tilde{T}_1}{C_1C_2}\right\}\sigma_2^2 - 2\left\{\lambda^2(\mu^+ + \mu^-)\chi_2 + \mu^+\chi_1 - 2(2\mu^+ + \mu^-)\frac{\tilde{T}_1}{C_1C_2}\right\}\sigma_2 \\ + \left\{(\mu^+)^2\chi_1 + \lambda^2(\mu^+ + \mu^-)^2\chi_2 - 4\mu^+(\mu^+ + \mu^-)\frac{\tilde{T}_1}{C_1C_2}\right\} = 0\ ,\end{aligned} \tag{6.8}$$

within which

$$\begin{aligned}\chi_1 &= \left(\tilde{T}_2 - T_2\right)\left(\lambda^{-2}T_1\tilde{T}_1 - \lambda^2\right) + 4\tilde{T}_1\ ,\\ \chi_2 &= \left(1 - T_2\tilde{T}_2\right)\left(\lambda^{-2}\tilde{T}_1 - \lambda^2 T_1\right) - 4\tilde{T}_1T_1T_2\ ,\\ \mu^+ &= C\left(\lambda^2 + \lambda^{-2}\right)\ , \quad \mu^- = C\left(\lambda^{-2} - \lambda^2\right)\ .\end{aligned}$$

In addition, it is also noted that p_1 and q_2 now take the particularly simple forms $p_1 = \lambda^{-2}$ and $q_1 = \lambda^2$, and that p_2 and q_2 are now equal to unity.

Figures 4(a), 4(b) and 4(c) about here.

In figure 4 three plots of the two neutral curves indicated by the solutions of equation (6.8) are presented showing the variation of σ_2 against kh for different values of λ and different values of the aspect ratio ξ. In figure 4(a) plots are presented for which $\xi = 1.0$

and $\lambda = 0.1, 0.3,$. The region in between the two curves associated with a specific value of λ indicates the region which is stable in the sense that the phase speed is real. The neutral curves indicate the values of σ_2 at which bifurcation from the homogeneous deformation occurs. Outside the domain of stability the use of a linear theory is highly questionable and one must then return to the governing equations and reformulate them as a non-linear problem. This technique has been applied to some simple model problems by Fu [24] and Fu and Rogerson [25]. A common feature of all the neutral curves shown in figure 4(a), 4(b) and 4(c) is that whenever the long wave ($kh, kd \to 0$) limit exits both values of σ_2 are finite. It has already been stated that this contrasts with the case for extensional waves, see Rogerson and Sandiford [16], in which only one value of σ_2 remains finite.

A glance at figures 4(b) and 4(c) indicates that the neutral curves coalesce at a point in the low kh regime for the values $\lambda = 0.1$ and $\lambda = 3.0$, respectively. The reason this occurs is that in these cases at some value of kh no real values of σ_2 exist for which standing wave solutions may be found. As this seems to occur at some low wave number it will be further investigated by considering the values of σ_2 in the long wave limit $kh, kd \to 0$. In this limit the appropriate values of σ_2, σ_2^0 say, may be obtained from equation (6.4), thus

$$(\xi + \lambda^4)\left(\sigma_2^0\right)^2 - 2C\lambda^2(1+\xi)\sigma_2^0 - C^2\left(\lambda^4 - 1\right)(1-\xi) = 0 \,. \tag{6.9}$$

It is clear from equation (6.9) that two real values of σ_2^0 exist in the case when $\xi = 1$, namely $\sigma_2^0 = 0$ and $\sigma_2^0 = \frac{4C\lambda^2}{1+\lambda^4}$, for all values of λ. In any further discussion concerning the existence of real roots of equations (6.9) we need consider separately the cases $\lambda > 1$ and $\lambda < 1$. (The unstrained case $\lambda = 1$ leads to the two real roots $\sigma_2^0 = 0$, $\sigma_2^0 = 2$.)

6.1.1 $\lambda > 1$

In the case when $\lambda > 1$ the two roots of equation (6.5) will always by real provided $\xi < 1$, however in the case $\xi > 1$ the two roots are not always real. An examination of the discriminant associated with (6.5) reveals that it vanishes if and only if

$$\lambda^4 = \frac{\xi + \xi^{\frac{1}{2}}}{\xi^{\frac{1}{2}} - 1} \,, \tag{6.10}$$

and accordingly we conclude that the two roots of (6.8) are real provided

$$\lambda^4 \leq \Theta(\xi) \,, \quad \Theta(\xi) = \frac{\xi + \xi^{\frac{1}{2}}}{\xi^{\frac{1}{2}} - 1} \,, \quad \xi = \frac{d}{h} > 1 \,. \tag{6.11}$$

where it is easily verified that the function $\Theta(\xi)$ has a local minimum given by $\Theta(\xi_{\min}) = (3 + 2\sqrt{(2)})$, ensuring that λ is always greater than 1. It is therefore concluded that if $\xi < 1$ and λ violates (6.11) then there exists a stable region in the low wave number regime. This region is highly stable in the sense that the phase speed remains real for all real values of σ_2.

6.1.2 $\lambda < 1$

When $\lambda < 1$ the two roots of equation (6.9) will always be real provided $\xi > 1$. In the case when $d < h$ real roots do not always exist and it may be shown that the existence of real roots may only be guaranteed provided

$$\lambda^4 \geq \Gamma(\xi) \,, \quad \Gamma(\xi) = \frac{\xi^{\frac{1}{2}} - \xi}{1 + \xi^{\frac{1}{2}}} \,, \quad \xi < 1. \tag{6.12}$$

It is noted that the maximum value of $\Gamma(\xi)$ is given by $\Gamma(\xi_{\max}) = (3 - 2\sqrt{2})$, indicating that λ is always less than 1. Again violation of the inequality (6.12) indicates the existence of a stable region within the low wave number region within which the phase speed is always real.

References

[1] M. A. Hayes and R. S. Rivlin, "Surface waves in deformed elastic materials," *Arch. ration. mech. Anal.*, vol. 8, pp. 358–380, 1961.

[2] J. N. Flavin, "Surface waves in pre-stressed Mooney material," *Q. Jl Mech appl. Mech*, vol. 16, pp. 441–449, 1963.

[3] A. J. Willson, "Surface and plate waves in biaxially-stressed elastic media," *Pure Appl. Geophys.*, vol. 102, pp. 182–192, 1973.

[4] A. J. Willson, "Surface plate waves in uniaxially-stressed Mooney material," *Pure Appl. Geophys.*, vol. 112, pp. 352–364, 1974.

[5] A. J. Willson, "The anomalous surface wave in uniaxially-stressed elastic material," *Pure Appl. Geophys.*, vol. 112, pp. 665–674, 1973.

[6] P. Chadwick and D. A. Jarvis, "Surface waves in a pre-stressed elastic body," *Proc. R. Soc. Lond. A*, vol. 336, pp. 517–536, 1979.

[7] M. A. Dowaikh and R. W. Ogden, "On surface waves and deformations in a pre-stressed incompressible elastic solid," *IMA J. of Appl. Math.*, vol. 44, pp. 261–284, 1990.

[8] P. Chadwick and D. A. Jarvis, "Interfacial waves in a pre-strained neo-Hookean body I. Biaxial states of strain," *Q. Jl mech. appl. Math*, vol. 32, pp. 387–399, 1979.

[9] P. Chadwick and D. A. Jarvis, "Interfacial waves in a pre-strained neo-Hookean body II. Triaxial states of strain," *Q. Jl mech. appl. Math*, vol. 32, pp. 401–418, 1979.

[10] M. A. Dowaikh and R. W. Ogden, "Interfacial waves and deformations in pre-stressed elastic media," *Proc. R. Soc. Lond. A.*, vol. 433, pp. 313–328, 1991.

[11] R. W. Ogden and D. A. Sotiropoulos, "On interfacial waves in pre-stressed layered incompressible solids," *Proc. R. Soc. Lond. A*, vol. 450, pp. 319–341, 1995.

[12] D. A. Sotiropoulos and C. G. Sifniotopoulos, "Interfacial waves in pre-stressed incompressible elastic interlayers," *J. Mech. Phys. Solids*, vol. 43(3), pp. 365–387, 1995.

[13] P. Chadwick, "Interfacial and surface waves in pre-strained isotropic elastic media," *Z angew Math Phys*, vol. 46 (special issue), pp. 51–71, 1995.

[14] P. M. Sheridan, F. O. James, and T. S. Miller, "Design of components," in *Engineering with rubber* (A. N. Gent, ed.), pp. 209–235, Munich:Hanser, 1992.

[15] G. A. Rogerson, "Waves in pre-stressed elastic laminates," in *Proceedings of IUTAM & ISIMM Symposium on Anisotropy, Inhomogeneity and Nonlinearity in Solid Mechanics*, pp. 475–480, Kluwer, 1995.

[16] G. A. Rogerson and K. J. Sandiford, "On small amplitude vibrations of pre-stressed laminates," *Int. J. Eng. Sci.*, vol. 34(8), pp. 853–872, 1996.

[17] G. A. Rogerson, "Some asymptotic expansions of the dispersion relation for an incompressible elastic plate," *Int. J. Solids Structures*, vol. 34(22), pp. 2785–2802, 1997.

[18] R. W. Ogden and D. G. Roxburgh, "The effect of pre-stress on the vibration and stability of elastic plates," *Int. J. Eng. Sci.*, vol. 30, pp. 1611–1639, 1993.

[19] G. A. Rogerson and Y. B. Fu, "An asymptotic analysis of the dispersion relation of a pre-stressed incompressible elastic plate," *Acta Mechanica*, vol. 111, pp. 59–77, 1995.

[20] W. A. Green, "Bending waves in strongly anisotropic elastic plates.," *Q. Jl Mech. Appl. Math*, vol. 35, pp. 485–507, 1982.

[21] E. R. Baylis and W. A. Green, "Flexural waves in fibre-reinforced laminated plates," *Journal of Sound and Vibration*, vol. 110(1), pp. 1–26, 1986.

[22] G. A. Rogerson, "Penetration of impact waves in a six-ply fibre composite laminate," *Jl. Sound and Vibr.*, vol. 158(1), pp. 105–120, 1992.

[23] F. Gilbert and G. E. Backus, "Propagator matrices in elastic wave and vibration problems," *Geophysics*, vol. 31, pp. 326–332, 1966.

[24] Y. B. Fu, "On the stability of inextensible elastic bodies:non-linear evolution of neutral, near neutral and non-neutral modes," *Proc. R. Soc. Lond. A.*, vol. 443, pp. 59–82, 1993.

[25] Y. B. Fu and G. A. Rogerson, "A non-linear analysis of instability of a pre-stressed incompressible elastic plate," *Proc. R. Soc. Lond. A.*, vol. 446, pp. 233–254, 1994.

Appendix A

The components of the propagator matrix $\mathbf{P}(h)$ are given by

$$
\begin{aligned}
P_{11} &= q_1 q_2 \{f(q_2)C_2 - f(q_1)C_1\}\mu^{-1} \ , \\
P_{12} &= q_1 q_2 \{q_1 f(q_2)S_1 - q_2 f(q_1)S_2\}\mu^{-1} \ , \\
P_{13} &= q_1 q_2 \{q_2 S_2 - q_1 S_1\}\mu^{-1} \ , \\
P_{14} &= q_1 q_2 \{C_1 - C_2\}\mu^{-1} \ , \\
P_{21} &= \{q_1 f(q_2)S_2 - q_2 f(q_1)S_1\}\mu^{-1} \ , \\
P_{22} &= q_1 q_2 \{f(q_2)C_1 - f(q_1)C_2\}\mu^{-1} \ , \\
P_{23} &= q_1 q_2 \{C_2 - C_1\}\mu^{-1} \ , \\
P_{24} &= \{q_2 S_1 - q_1 S_2\}\mu^{-1} \ , \\
P_{31} &= \{q_1 f(q_2)^2 S_2 - q_2 f(q_1)^2 S_1\}\mu^{-1} \ , \\
P_{32} &= q_1 q_2 f(q_1) f(q_2)\{C_1 - C_2\}\mu^{-1} \ , \\
P_{33} &= q_1 q_2 \{f(q_2)C_2 - f(q_1)C_1\}\mu^{-1} \ , \\
P_{34} &= \{q_2 f(q_1)S_1 - q_1 f(q_2)S_2\}\mu^{-1} \ , \\
P_{41} &= q_1 q_2 f(q_1) f(q_2)\{C_2 - C_1\}\mu^{-1} \ , \\
P_{42} &= q_1 q_2 \{q_1 f(q_2)^2 S_1 - q_2 f(q_1)^2 S_2\}\mu^{-1} \ , \\
P_{43} &= q_1 q_2 \{q_2 f(q_1)S_2 - q_1 f(q_2)S_1\}\mu^{-1} \ , \\
P_{44} &= q_1 q_2 \{f(q_2)C_1 - f(q_1)C_2\}\mu^{-1} \ ,
\end{aligned}
$$

within which $S_m = \text{Sinh}\, kq_m h$, $C_m = \text{Cosh}\, kq_m h$ and $\mu = q_1 q_2 \gamma (q_2^2 - q_1^2)$.

Appendix B

The order one quantities A_m, occurring in equation (4.34), are given by

$$
\begin{aligned}
A_1 &= a_0 d_1^{(0)} f^{(0)^2}, \\
A_2 &= \left(2a_0\xi - \frac{1}{4a_0}\right) d_1^{(0)} f^{(0)^2} + a_0 d_1^{(2)} f^{(0)^2} + a_0 d_1^{(0)} \left(2f^{(0)} f^{(2)} - f^{(1)^2}\right), \\
A_3 &= \left\{2\sqrt{a_0} f^{(0)} f^{(1)} + \frac{f^{(0)^2}}{2\sqrt{a_0}}\right\} \left\{\sqrt{a_0} d_3^{(1)} - \frac{d_1^{(0)}}{2\sqrt{a_0}}\right\}, \\
A_4 &= \left\{2\sqrt{a_0} f^{(0)} f^{(1)} + \frac{f^{(0)^2}}{2\sqrt{a_0}}\right\} \left\{\left(p_1^{(0)} p_2^{(0)} - a_0\right)\left(\tilde{f}_1^{(0)} - \tilde{f}_2^{(0)}\right) f^{(1)}\right\}, \\
A_5 &= 2\sqrt{a_0} d_1^{(0)} f^{(0)^2}\xi - a_0 d_3^{(2)} f^{(0)^2} - \frac{d_3^{(1)} f^{(0)^2}}{2} \\
&\quad + \sqrt{a_0} d_1^{(0)} \left\{a_0 f^{(1)^2} + 2\sqrt{a_0} f^{(0)} f^{(1)} + \frac{f^{(0)} f^{(1)}}{\sqrt{a_0}}\right\}, \\
A_6 &= \sqrt{a_0} f^{(0)^2} f^{(1)} \left(a_0 + p_1^{(0)} p_2^{(0)}\right)\left(\tilde{f}_2^{(0)} - \tilde{f}_1^{(0)}\right),
\end{aligned}
$$

within which $d_i^{(m)}$ represents the coefficient of b^m in d_i, $f^{(m)}$ the coefficients of b^m in $f(q)$, p_1 and p_2 have been expanded to give $p_1 = p_1^{(0)} + p_1^{(2)} b^2$ and $p_2 = p_2^{(0)} + p_2^{(2)} b^2$, where

$$
\begin{aligned}
d_1^{(0)} &= p_2^{(0)} \left(f^{(0)} - \tilde{f}_1^{(0)}\right)^2 - p_1^{(0)} \left(f^{(0)} - \tilde{f}_2^{(0)}\right)^2, \\
d_1^{(2)} &= 2p_1^{(0)} \left(f^{(0)} - \tilde{f}_2^{(0)}\right)\left(\tilde{f}_2^{(0)} - f^{(2)}\right) + 2p_2^{(0)} \left(f^{(0)} - \tilde{f}_1^{(0)}\right)\left(f^{(2)} - \tilde{f}_1^{(2)}\right) \\
&\quad + f^{(1)^2} \left(p_1^{(0)} - p_2^{(1)}\right) + p_2^{(2)} \left(f^{(0)} - \tilde{f}_1^{(0)}\right)^2 - p_1^{(2)} \left(f^{(0)} - \tilde{f}_2^{(2)}\right)^2, \\
d_2^{(1)} &= 2f^{(1)} \left\{p_1^{(0)} \left(f^{(0)} - \tilde{f}_2^{0}\right) - p_2^{(0)} \left(f^{(0)} - \tilde{f}_1^{(0)}\right)\right\}, \\
d_2^{(2)} &= 2f^{(1)^2} \left(p_1^{(0)} - p_2^{(0)}\right) - d_1^{(2)}, \\
f^{(0)} &= \gamma(1 - a_0) - \sigma_2, \qquad f^{(1)} = \gamma, \qquad f^{(2)} = -\gamma a_1, \\
\tilde{f}_1^{(0)} &= \tilde{\gamma} + \frac{\lambda_0 + \mu_0}{2} - \sigma_2, \qquad \tilde{f}_1^{(2)} = \frac{\mu_2}{4\mu_0} - \gamma a_1 \frac{\tilde{\rho}}{\rho}, \\
\tilde{f}_2^{(0)} &= \tilde{\gamma} + \frac{\lambda_0 - \mu_0}{2} - \sigma_2, \qquad \tilde{f}_2^{(2)} = -\frac{\mu_2}{4\mu_0} - \gamma a_1 \frac{\tilde{\rho}}{\rho}, \\
p_1^{(0)} &= \left(\frac{\lambda_0 + \mu_0}{2\tilde{\gamma}}\right)^{\frac{1}{2}}, \qquad p_1^{(2)} = \frac{\mu_2/2\mu_0 - 2\gamma a_1 \tilde{\rho}/\rho}{2(2\tilde{\gamma})^{\frac{1}{2}}(\lambda_0 + \mu_0)^{\frac{1}{2}}}, \\
p_2^{(0)} &= \left(\frac{\lambda_0 - \mu_0}{2\tilde{\gamma}}\right)^{\frac{1}{2}}, \qquad p_2^{(2)} = -\frac{\mu_2/2\mu_0 + 2\gamma a_1 \tilde{\rho}/\rho}{2(2\tilde{\gamma})^{\frac{1}{2}}(\lambda_0 + \mu_0)^{\frac{1}{2}}}, \\
\lambda_0 &= 2\tilde{\beta} - \frac{\tilde{\rho}}{\rho}(2\beta + 2\gamma a_0), \\
\mu_0 &= \left\{4(\tilde{\beta}^2 - \tilde{\alpha}\tilde{\gamma}) + \frac{\tilde{\rho}^2}{\rho^2}(2\beta + 2\gamma a_0)^2 + 4(\tilde{\gamma} - \tilde{\beta})\frac{\tilde{\rho}}{\rho}\right\}^{\frac{1}{2}}, \\
\mu_2 &= 4\frac{\tilde{\rho}^2}{\rho^2}(2\beta + 2\gamma a_0)\gamma a_1 + 8\frac{\tilde{\rho}}{\rho}(\tilde{\gamma} - \tilde{\beta})\gamma a_1.
\end{aligned}
$$

Figure 1: Numerical solutions of the dispersion relation for $\alpha = 3.0$, $2\beta = 4.5$, $\gamma = 0.5$, $\tilde{\alpha} = 1.0$, $2\tilde{\beta} = 2.5$, $\tilde{\gamma} = 1.2$, $\sigma_2 = 1.2$, $v_R = 1.596$, $v_I = -$, $v_L = 1.732$ and $\tilde{v}_L = 1.0$. (a) Fundamental mode and first fifteen harmonics and (b) Group velocity associated with first seven harmonics

Figure 2: Numerical solutions of the dispersion relation for $\alpha = 4.0$, $2\beta = 5.0$, $\gamma = 1.0$, $\tilde{\alpha} = 3.0$, $2\tilde{\beta} = 1.5$, $\tilde{\gamma} = 0.5$, $\sigma_2 = 1.2$, $v_R = 1.999$, $v_I = -$, $v_L = 2.0$ and $\tilde{v}_L = 1.581$. (a) Fundamental mode and first five harmonics, (b) first four harmonics showing asymptotic expansions in the shortwave limit to second order, (c) to third order.

Figure 3: Numerical solutions of the dispersion relation for $\alpha = 3.0$, $2\beta = 1.5$, $\gamma = 0.5$, $\tilde{\alpha} = 4.0$, $2\tilde{\beta} = 5.0$, $\tilde{\gamma} = 1.0$, $\sigma_2 = 1.2$, $v_R = 0.901$, $v_I = -$, $v_L = 1.581$ and $\tilde{v}_L = 2.0$. (a) Fundamental mode and first five harmonics, (b) first four harmonics showing asymptotic expansions in the shortwave limit to second order, (c) to third order.

Figure 4: Neutral curves showing σ_2 against kh for $\lambda = 0.1$ and 3.0. (a) $\zeta = 1.0$, (b) $\zeta = 2.0$ and (c) $\zeta = 0.5$.

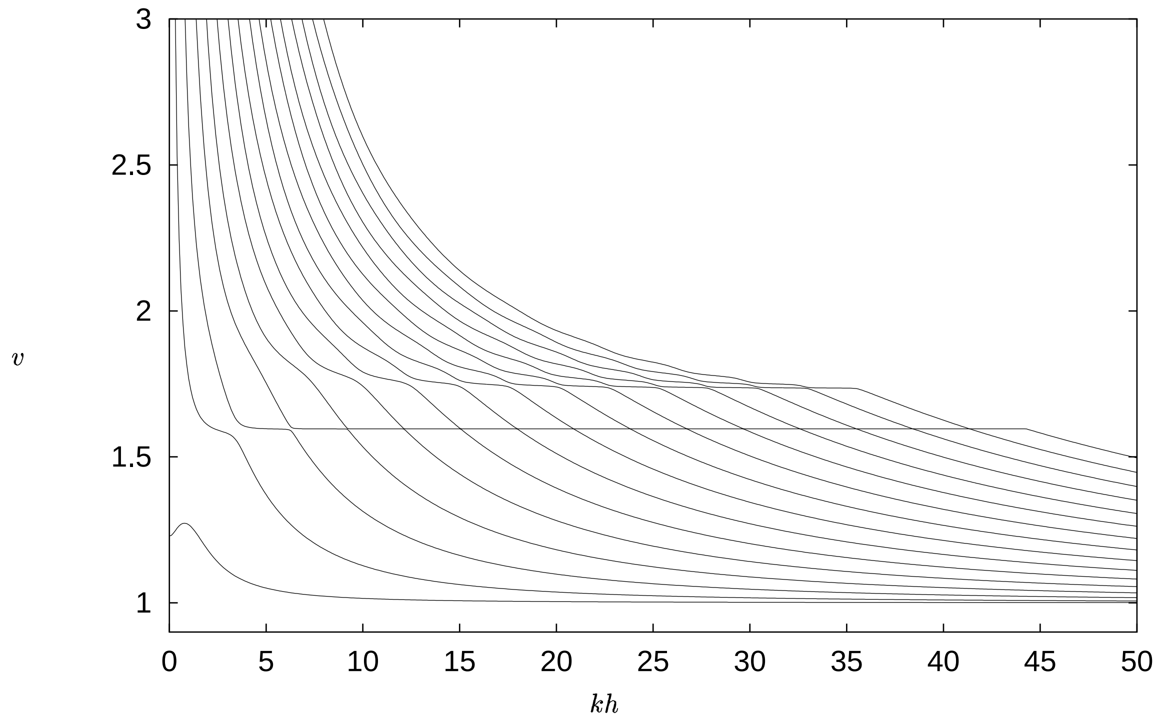

Figure 1(a)

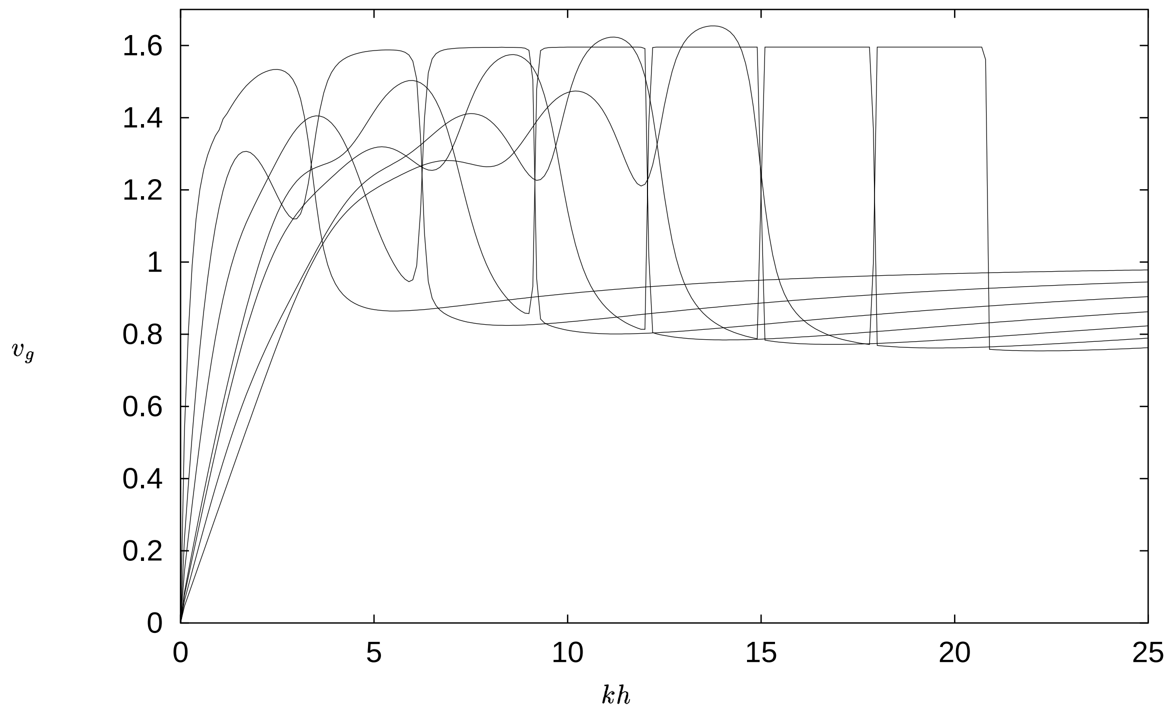

Figure 1(b)

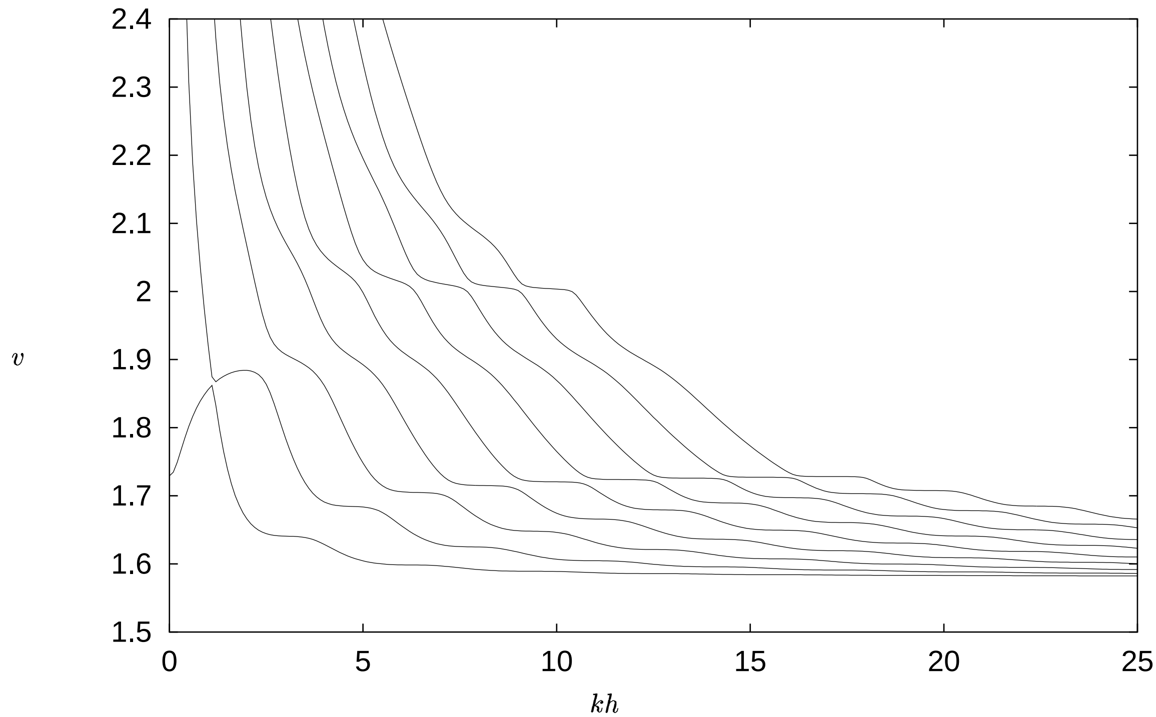

Figure 2(a)

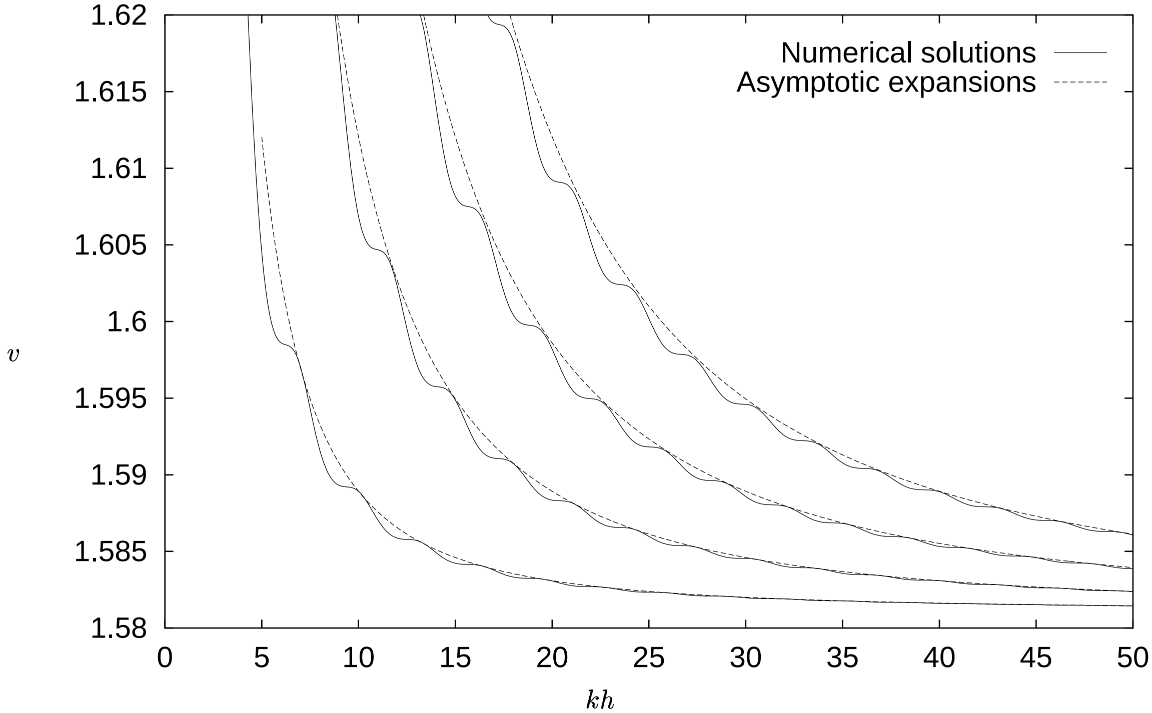
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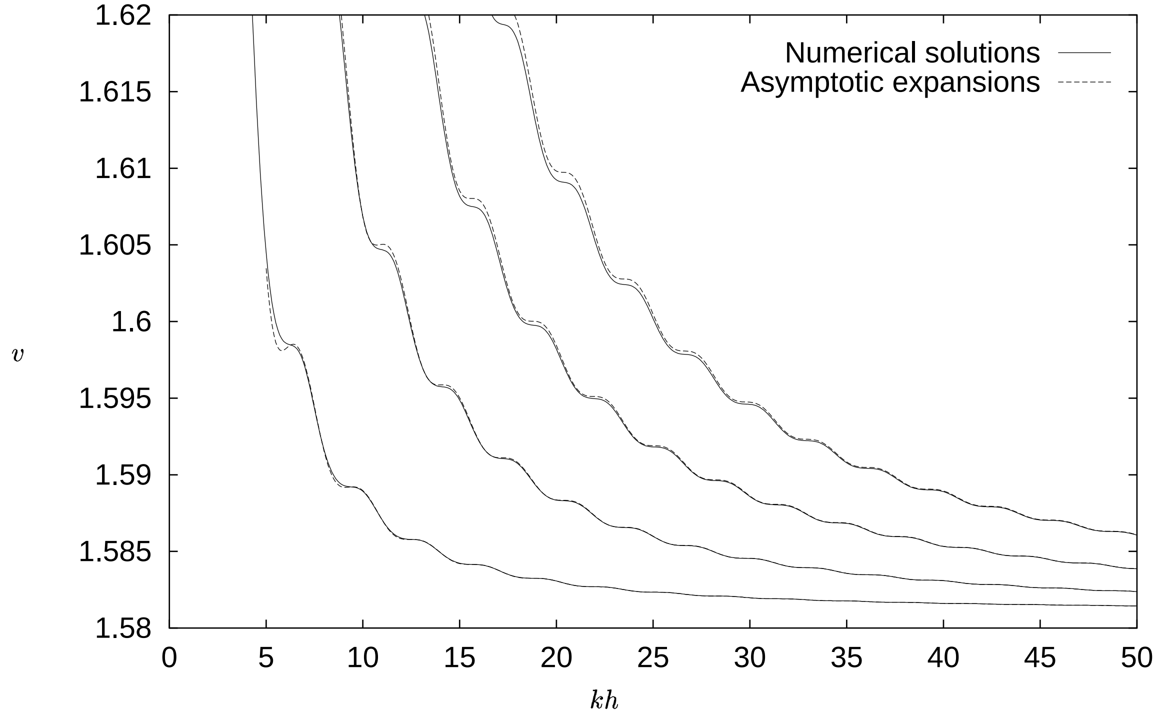
Figure 2(b)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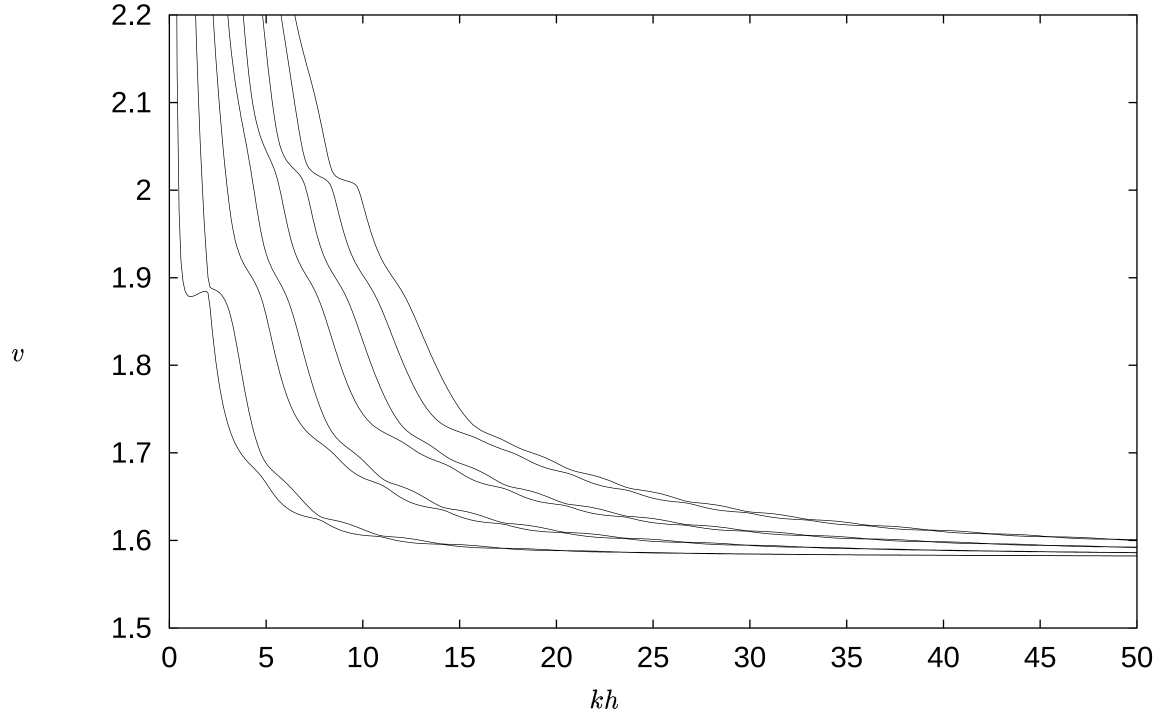

Figure 3(a)

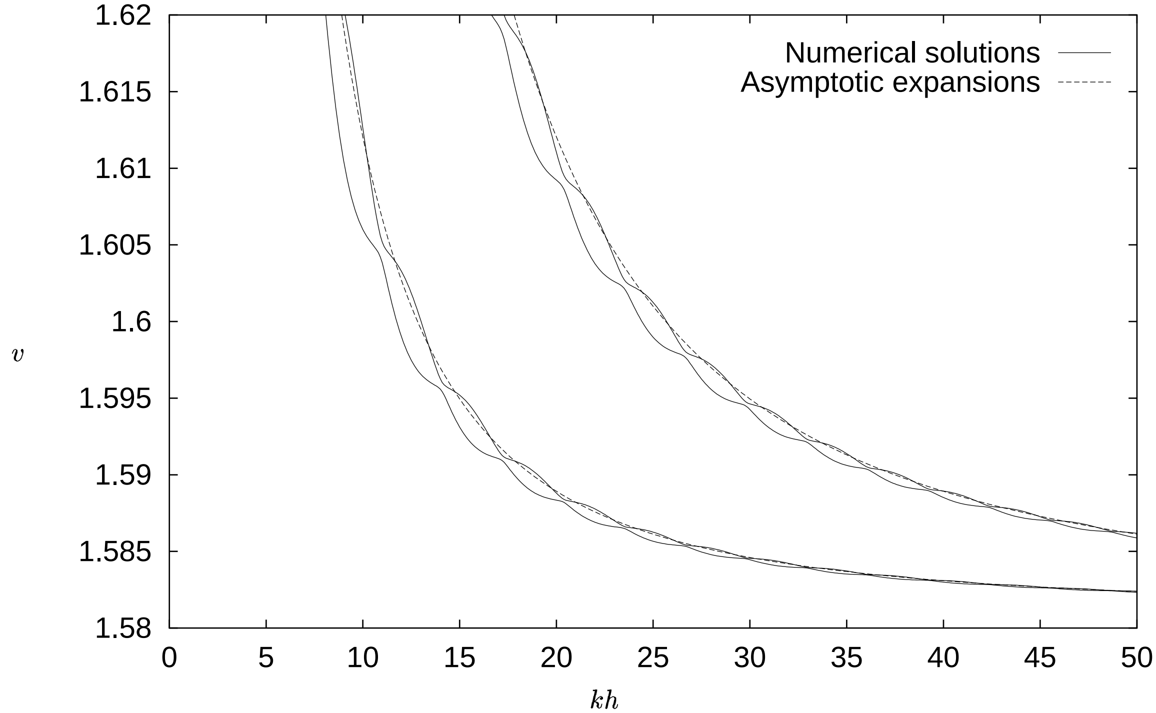
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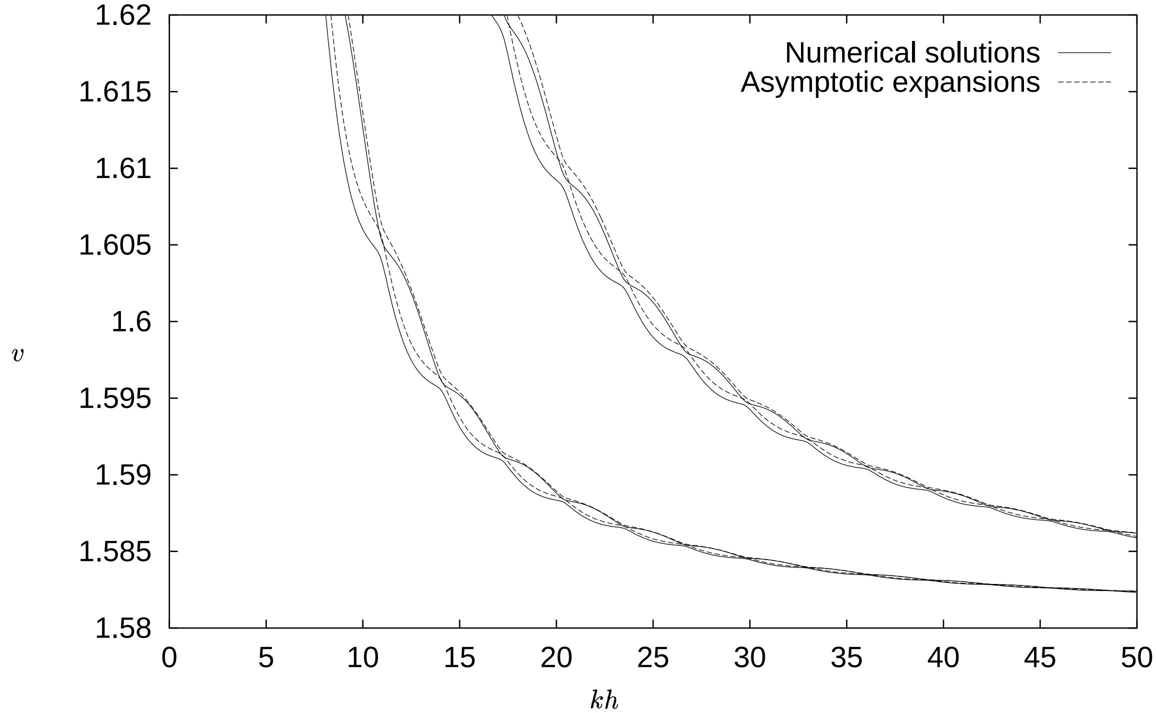
Figure 3(b)



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1.605
1.6
1.595
1.59
1.585
1.58
v
0
5
10
15
20
25
30
35
40
45
50
kh
Numerical solutions
Asymptotic expansions

Figure 3(c)

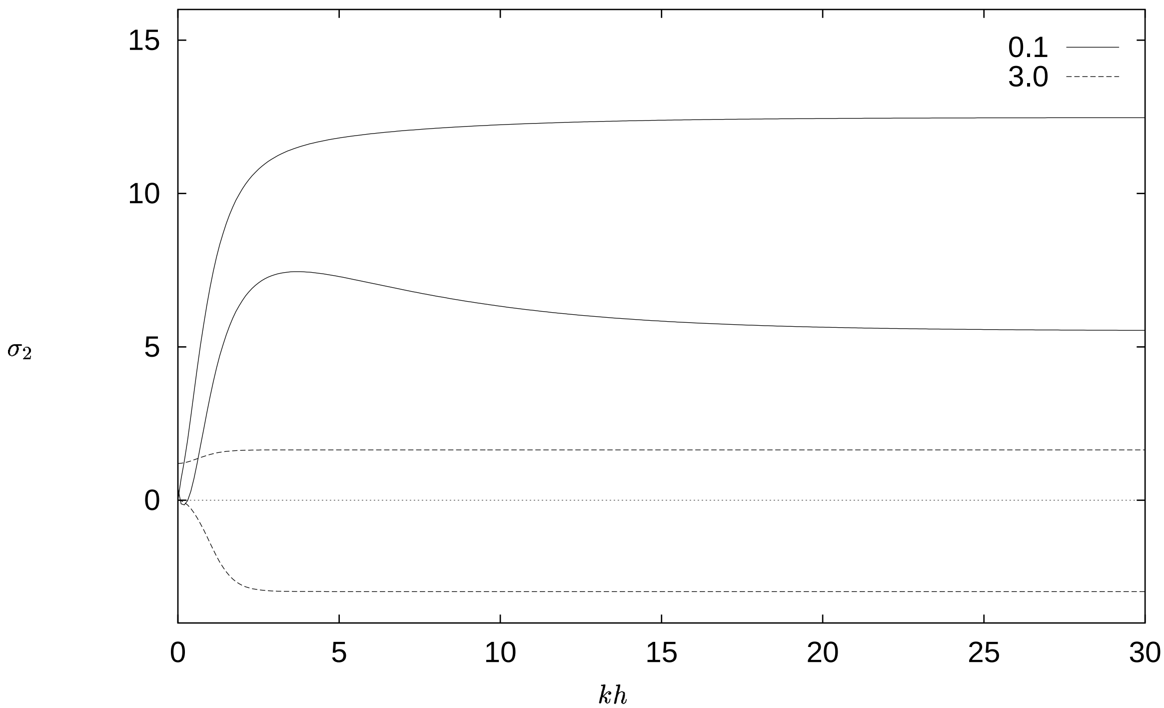

Figure 4(a)

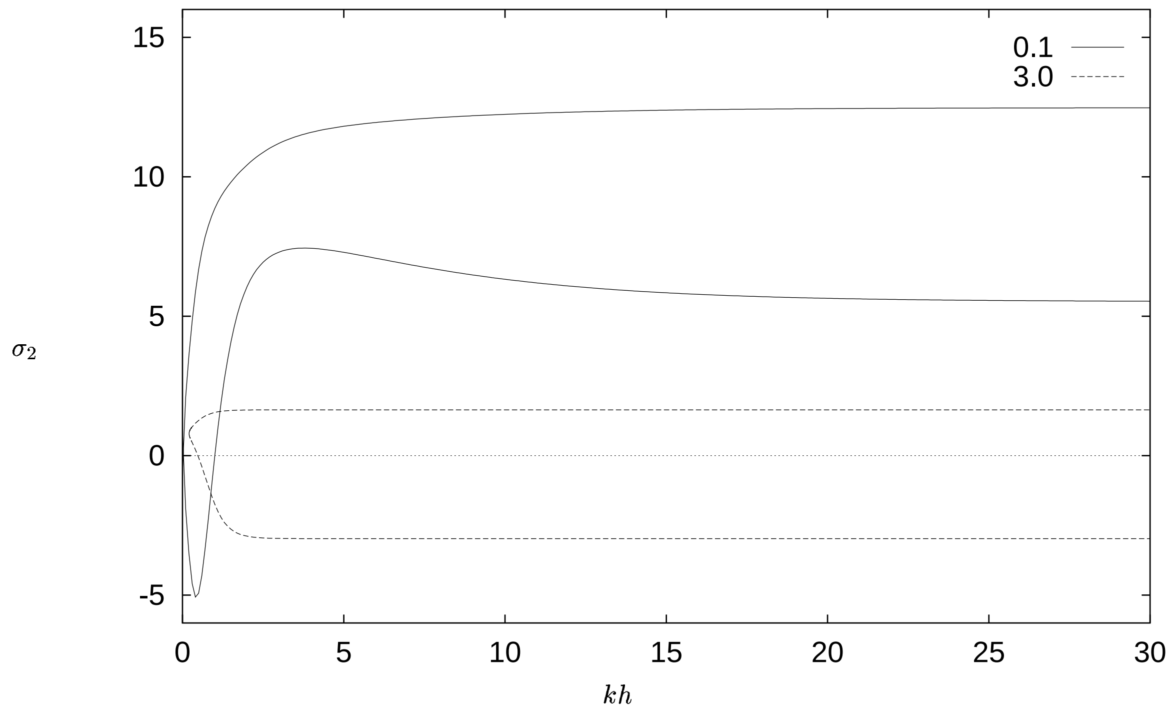

Figure 4(b)

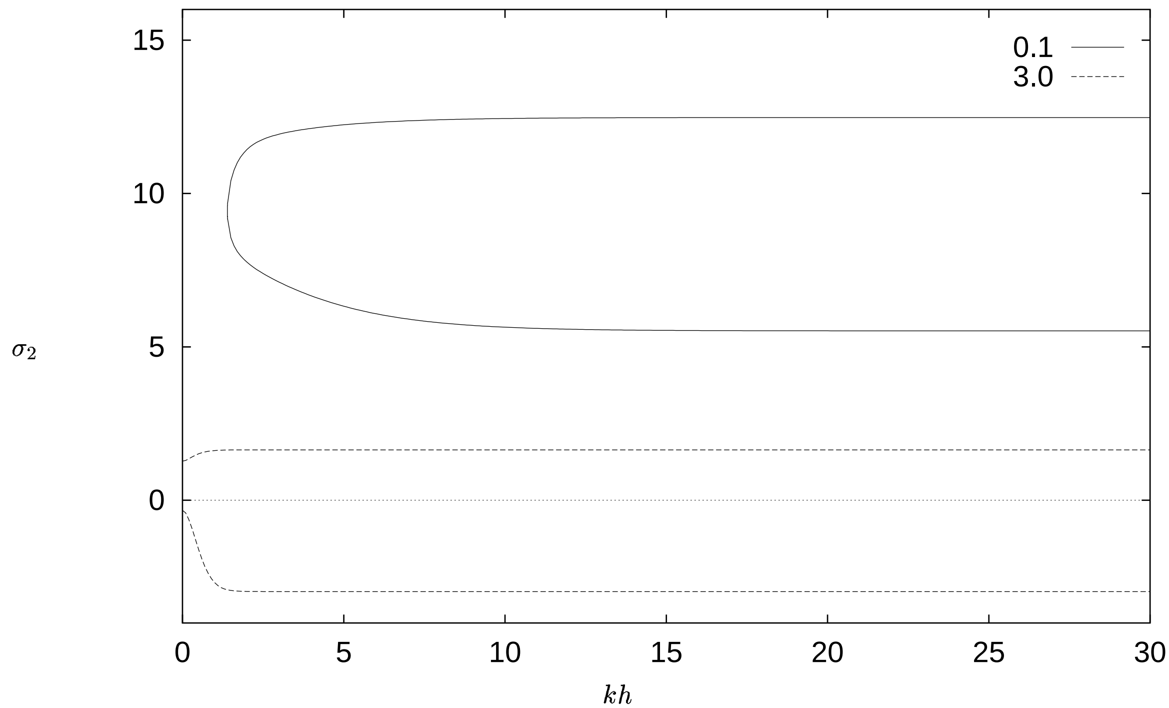

Figure 4(c)