Harmonic wave propagation along a non-principal direction in a pre-stressed elastic plate

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Abstract

In this paper the problem of harmonic wave propagation in a pre-stressed, incompressible elastic plate is investigated. Specifically the situation where one principal direction of the primary deformation is normal to the plate and the direction of propagation is at an angle θ to one of the in-plane principal direction is considered. The dispersion relation is derived in respect of a general strain energy function and numerical solutions presented for a Mooney-Rivlin material. The dispersion curves are shown to be more complex in nature than those associated with propagation along a principal direction, as the solution arising from the horizontally polarised shear wave does not uncouple. An asymptotic analysis, for high and low wave number, is carried out with the high wave number expansions providing a good approximation to the numerical solution over a large wave number region. The paper also includes an investigation of the effects of changes in the normal Cauchy stress and direction of propagation on the existence of surface waves.

1 Introduction

Primarily motivated by the increasing industrial application of laminated structures, theoretical study of wave propagation and vibration in layered media has been an area of considerable recent research activity. Within the framework of linear elastodynamics there has been a huge number of contributions,an extensive list of references may be found in the recent review article by Chimenti [1]. In comparison, there have been surprisingly few investigations aimed at elucidating the effects of pre-stress on the dynamic characteristics of layered media. When a solid is subject to pre-stress, either arising in manufacture or by the action of external forces, its dynamic behaviour can be greatly effected. In particular, the underlying pre-stress may be of a form as to make the structure unstable with respect to small amplitude time dependent superimposed motions.

The first attempts to examine the effects of pre-stress on wave motion in bounded and semi-infinite media where seemingly in the context of surface waves, see e.g. Hayes and Rivlin [2] and Favin [3]. More recent contributions, within the same context, were made by Chadwick and Jarvis [4] and Dowaikh and Ogden [5]. Readers are referred to this latter paper for a detail list of early references. In the case of an elastic plate several recent studies have considered the plane strain problem involving propagation along a principal direction in a plate composed of incompressible elastic, see e.g. Ogden and Roxburgh [6], Rogerson and Fu [7] and Rogerson [8]. Additionally, the problems of flexural and extensional waves have been considered in 4-ply incompressible laminated plates, see Rogerson and Sandiford [9] and Rogerson and Sandiford [10]. The effects of shear have also been investigated by Connor and Ogden [11] and Connor and Ogden [12] in the contexts of surface waves and plate waves, respectively. In this case the direction of propagation does not coincide with one of the principal directions.

In this paper the effect of pre-stress on the propagation in small amplitude waves in an incompressible elastic plate is considered. In particular one of the principal axes of primary deformation are normal to the surface of the plate and the direction of propagation is at an angle θ to one of the in-plane principal directions. In addition to yielding more information concerning the effect of pre-stress on dynamic response, it is envisaged that the results will have relevance to layered media in which all layers have only one common normal principal direction. Additionally, to further motivate the present study we cite the increasing use of rubber-like components. The industrial application of such components is widespread, including engine mounts, off-shore structure flex joints and vibration insulators. In particular, it is noted that the application of rubber-like components as vibration insulators in bridges and tall buildings has specific relevance to earthquake protection, see Seridan et.al [13].

In section 2 of this paper the governing equations are derived and the dispersion relations associated with flexural and extensional waves are derived. In section 3 some numerical solutions, giving phase speed as a function of wave number, are presented in respect of a Mooney-Rivlin strain energy function. It is observed, in contrast to the plane strain case, that there exists three finite long wave limits. In the short wave limit the harmonics tend to the lower of two associated shear wave speeds and the fundamental mode tends to the associated surface wave speed. The existence of this surface wave speed is dependent on pre-stress and when it does not exist the fundamental mode tends to the appropriate shear wave speed. A further feature observed numerically is that harmonics associated with either flexural or extensional waves intersect. In the plane strain case this phenomenon does not occur. By considering propagation along one of the in-plane principal directions it is establishes that similar behaviour occurs if the dispersion relation associated with the horizontally polarised shear waves is included. In the plane strain case the solution corresponding to this wave uncouples from that arising from the two other displacement component.

In section 4 the dispersion relation is investigated analytically in both the high wand low wave number regions. Equations are derived for the long wave limit of the fundamental modes and all corresponding finite limits of the harmonics. In the high wave number region one possible limit of the fundamental mode is the surface wave speed. The corresponding equation is derived and the existence of such waves investigated both in respect of changes in the normal Cauchy stress and angle of propagation. The paper is concluded with the derivation of high wave number approximations for the phase speed associated with each harmonic. These are shown to provide good agreement with the numerical solution.

2 Derivation of governing equations

Consider a single layer plate of width $2h$ which is formed of a pre-stressed incompressible elastic material of finite thickness and infinite lateral extent. An appropriate Cartesian coordinate system is chosen coincident with the principal axes of the right Cauchy-Green strain tensor in a pre-stressed equilibrium state B_e, such that the origin O lies in the mid-plane and Ox_2 is normal to the plane of the plate. The linearised form of the equation of motion for a pre-stressed incompressible material has previously been derived, see for example Dowaikh and Ogden [5], and takes the form

$$B_{jilk}u_{k,lj} - p^*_{,i} = \rho\ddot{u}_i, \tag{2.1}$$

in which B_{ijkl} are components of the elasticity tensor, $\mathbf{u}$ the infinitesimal displacement, ρ the material density, p^* a time dependent pressure increment and a comma and dot indicate differentiation with respect to $\mathbf{x}$ and time, respectively. The three components of the linearised equation of motion (2.1) may be written explicitly as

$$B_{1111}u_{1,11} + (B_{1122} + B_{2112})u_{2,12} + B_{2121}u_{1,22} \\ + (B_{1133} + B_{3113})u_{3,13} + B_{3131}u_{1,33} - p^*_{,1} = \rho\ddot{u}_1, \tag{2.2}$$

$$(B_{1221} + B_{2211})u_{1,12} + B_{1212}u_{2,11} + B_{2222}u_{2,22} \\ + (B_{2233} + B_{3223})u_{3,32} + B_{3232}u_{2,33} - p^*_{,2} = \rho\ddot{u}_2, \tag{2.3}$$

$$(B_{1331} + B_{3311})u_{1,13} + (B_{2332} + B_{3322})u_{2,23} + B_{1313}u_{3,11} \\ + B_{2323}u_{3,22} + B_{3333}u_{3,33} - p^*_{,3} = \rho\ddot{u}_3. \tag{2.4}$$

A measure of linearised traction increments has also been previously obtained in the form

$$\tau_i = (B_{jilk}u_{k,l} + \bar{p}u_{j,i} - p^*\delta_{ij})\, n_j, \tag{2.5}$$

in which $\bar{p}$ is a static pressure in B_e and $\mathbf{n}$ is the unit outward normal to a material surface in B_e, see e.g. Dowaikh and Ogden [5]. For the plate in question $\mathbf{n} = \delta_{i2}$ and the components of incremental surface traction are given explicitly by

$$\tau_1 = (B_{2112} + \bar{p})u_{2,1} + B_{2121}u_{1,2}, \tag{2.6}$$

$$\tau_2 = B_{2211}u_{1,1} + (B_{2222} + \bar{p})u_{2,2} + B_{2233}u_{3,3} - p^*, \tag{2.7}$$

$$\tau_3 = (B_{2332} + \bar{p})u_{2,3} + B_{2323}u_{3,2}. \tag{2.8}$$

The equations of motion (2.2)–(2.4) must be solved in conjunction with the linearised incompressibility constraint $u_{i,i} = 0$. Solutions of these equations are now sought in the form of the travelling wave

$$(u_1, u_2, u_3, p^*) = (U, V, W, kP)e^{kqx_2}e^{ik(x_1 \sin\theta + x_3 \cos\theta - vt)}, \tag{2.9}$$

in which k is the wave number, v is the phase speed, θ is the angle of propagation relative to the Ox_3 axis and q is to be determined. If solutions of the form indicated in equation (2.9) are inserted into equations (2.2)–(2.4) we obtain the three equations of motion

$$U(q^2B_{2121} - B_{1111}\sin^2\theta - B_{3131}\cos^2\theta + \rho v^2) + iV(B_{1122} + B_{2112})q\sin\theta$$
$$-W(B_{1133} + B_{3113})\sin\theta\cos\theta - iP\sin\theta = 0, \tag{2.10}$$

$$iU(B_{1221} + B_{2211})q\sin\theta + V(q^2B_{2222} - B_{1212}\sin^2\theta - B_{3232}\cos^2\theta + \rho v^2)$$
$$+iW(B_{2233} + B_{3223})q\cos\theta - qP = 0, \tag{2.11}$$

$$-U(B_{1331} + B_{3311})\sin\theta\cos\theta + iV(B_{2332} + B_{3322})q\cos\theta$$
$$+W(B_{2323}q^2 - B_{1313}\sin^2\theta - B_{3333}\cos^2\theta + \rho v^2) - iP\cos\theta = 0, \tag{2.12}$$

and the linearised incompressibility constraint

$$i\sin\theta U + qV + i\cos\theta W = 0. \tag{2.13}$$

On making use of the incompressibility constraint to eliminate V in favour of U and W within equations (2.10), (2.11) and (2.12), three simultaneous equations in three unknowns are obtained which possess non-trivial solutions provided

$$\gamma_{21}\gamma_{23}q^6 + q^4\{(\gamma_{21} + \gamma_{23})\rho v^2 - \mu_1\} + q^2(\rho^2v^4 - \mu_2\rho v^2 + \mu_3)$$
$$+ (\mu_4 - \rho v^2)(\rho v^2 - \mu_5) = 0, \tag{2.14}$$

within which

$$
\begin{aligned}
\mu_1 &= \gamma_{21}(\gamma_{13}\sin^2\theta + 2\beta_{23}\cos^2\theta) + \gamma_{23}(\gamma_{31}\cos^2\theta + 2\beta_{12}\sin^2\theta) \\
\mu_2 &= (\gamma_{23} + \gamma_{13} + 2\beta_{12})\sin^2\theta + (\gamma_{21} + \gamma_{31} + 2\beta_{23})\cos^2\theta, \\
\mu_3 &= (\gamma_{12}\sin^2\theta + \gamma_{32}\cos^2\theta)(\gamma_{23}\sin^2\theta + \gamma_{21}\cos^2\theta) + 2\beta_{12}\gamma_{13}\sin^4\theta \\
&\quad + 2\beta_{23}\gamma_{31}\cos^4\theta + \sin^2\theta\cos^2\theta\big\{\gamma_{13}\gamma_{31} + 4\beta_{13}\beta_{23} - \delta^2\big\}, \\
\mu_4 &= \gamma_{12}\sin^2\theta + \gamma_{32}\cos^2\theta, \\
\mu_5 &= \gamma_{13}\sin^4\theta + \gamma_{31}\cos^4\theta + 2\beta_{13}\sin^2\theta\cos^2\theta,
\end{aligned}
\tag{2.15}
$$

and

$$
\gamma_{ij} = B_{ijij}, \qquad 2\beta_{ij} = B_{iiii} + B_{jjjj} - 2B_{iijj} - 2B_{ijji},
$$
$$
\delta = \beta_{12} - \beta_{13} - \beta_{23}, \qquad \epsilon = \beta_{23} - \beta_{12} - \beta_{13},
$$

where ϵ is defined here for future use. If the roots of equation (2.14) are denoted by q_1^2, q_2^2 and q_3^2 the solutions of U, V, W and P are obtainable as linear combinations of the six linearly independent solutions, thus

$$
\begin{aligned}
U &= \sum_{m=1}^{3} U^{(2m-1)}E^{+} + U^{(2m)}E_m^{-}, \qquad & V &= \sum_{m=1}^{3} V^{(2m-1)}E_m^{+} + V^{(2m)}E_m^{-}, \\
W &= \sum_{m=1}^{3} W^{(2m-1)}E_m^{+} + W^{(2m)}E_m^{-}, \qquad & P &= \sum_{m=1}^{3} P^{(2m-1)}E_m^{+} + P^{(2m)}E_m^{-},
\end{aligned}
\tag{2.16}
$$

where $E_m^+ = \exp(kq_mx_2)$, $E_m^- = \exp(-kq_mx_2)$ $(m = 1,2,3)$, and $U^{(i)}$, $V^{(i)}$, $W^{(i)}$ and $P^{(i)}$ $(i = 1,2,\dots,6)$ are disposable constants. The four sets of six disposable constants for $U^{(i)}$, $V^{(i)}$, $W^{(i)}$ and $P^{(i)}$ are not independent as they are linked through the equations of motion and the incompressibility constraint. By making use of equations (2.10), (2.11), (2.12) and the incompressibility constraint (2.13) we may obtain

$$
\begin{aligned}
U &= \sum_{m=1}^{3} \frac{iq_m f(q_m,\rho v^2)}{\zeta(q_m,\rho v^2)}\big(V^{(2m-1)}E_m^{+} - V^{(2m)}E_m^{-}\big)\sin\theta, \\
V &= \sum_{m=1}^{3} \big(V^{(2m-1)}E_m^{+} + V^{(2m)}E_m^{-}\big), \\
W &= \sum_{m=1}^{3} \frac{iq_m h(q_m,\rho v^2)}{\zeta(q_m,\rho v^2)}\big(V^{(2m-1)}E_m^{+} - V^{(2m)}E_m^{-}\big)\cos\theta, \\
P &= \sum_{m=1}^{3} \frac{q_m g(q_m,\rho v^2)}{\zeta(q_m,\rho v^2)}\big(V^{(2m-1)}E_m^{+} - V^{(2m)}E_m^{-}\big),
\end{aligned}
\tag{2.17}
$$

where

$$
\begin{aligned}
f(q_m, \rho v^2) =& \gamma_{23} q_m^2 - \gamma_{13} \sin^2\theta + \delta \cos^2\theta + \rho v^2, \\
h(q_m, \rho v^2) =& \gamma_{21} q_m^2 + \epsilon \sin^2\theta - \gamma_{31} \cos^2\theta + \rho v^2, \\
g(q_m, \rho v^2) =& f(q_m, \rho v^2)\{\gamma_{21} q_m^2 + \sin^2\theta (B_{1122} + B_{2112} - B_{1111}) - \gamma_{31}\cos^2\theta + \rho v^2\} \\
& + h(q_m, \rho v^2)\cos^2\theta (B_{1122} + B_{2112} - B_{1133} - B_{3113}), \\
\zeta(q_m, \rho v^2) =& q_m^2(\gamma_{23}\sin^2\theta + \gamma_{21}\cos^2\theta) + \rho v^2 - \mu_5.
\end{aligned}
$$

Note that in the above derivations it is implicitly assumed that $\sin\theta \neq 0$ and $\cos\theta \neq 0$. Similarly, explicit representations of the traction increments may be found by inserting equation (2.17) into equations (2.6)–(2.8), thus

$$
\begin{aligned}
\frac{\tau_1}{ik} &= \sum_{m=1}^{3} \frac{F(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} \left\{ V^{(2m-1)} E_m^{+} + V^{(2m)} E_m^{-} \right\} \sin\theta, \\
\frac{\tau_2}{k} &= \sum_{m=1}^{3} \frac{q_m G(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} \left\{ V^{(2m-1)} E_m^{+} - V^{(2m)} E_m^{-} \right\}, \\
\frac{\tau_3}{ik} &= \sum_{m=1}^{3} \frac{H(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} \left\{ V^{(2m-1)} E_m^{+} + V^{(2m)} E_m^{-} \right\} \cos\theta,
\end{aligned}
\tag{2.18}
$$

within which

$$
\begin{aligned}
F(q_m, \rho v^2) =& \zeta(q_m, \rho v^2)(\gamma_{21} - \sigma_2) + \gamma_{21} q_m^2 f(q_m, \rho v^2), \\
G(q_m, \rho v^2) =& -\gamma_{21}\gamma_{23} q_m^4 + \sin^2\theta\cos^2\theta(\delta\gamma_{21} + \epsilon\gamma_{23} + \delta^2 - 4\beta_{13}\beta_{23} + 2\beta_{23}\sigma_2) \\
& + q_m^2 \Big\{ \gamma_{23}\big\{(2\beta_{12} + \gamma_{21} - \sigma_2)\sin^2\theta + \gamma_{31}\cos^2\theta - \rho v^2\big\} \\
& \qquad + \gamma_{21}\big\{(2\beta_{23} + \gamma_{23} - \sigma_2)\cos^2\theta + \gamma_{13}\sin^2\theta - \rho v^2\big\} \Big\} \\
& + (\rho v^2 - \gamma_{13})(\gamma_{31} - \rho v^2) + \sin^2\theta(\rho v^2 - \gamma_{13}\sin^2\theta)(2\beta_{12} + \gamma_{21} - \sigma_2) \\
& + \cos^2\theta(\rho v^2 - \gamma_{13}\cos^2\theta)(2\beta_{23} + \gamma_{23} - \sigma_2), \\
H(q_m, \rho v^2) =& \zeta(q_m, \rho v^2)(\gamma_{23} - \sigma_2) + \gamma_{23} q_m^2 h(q_m, \rho v^2).
\end{aligned}
$$

The dispersion relation associated with the plate is now derived from equations (2.18) by applying traction free boundary conditions on each free surface, namely $\tau_1 = \tau_2 = \tau_3 = 0$ on $x_2 =$

$\pm h$, thus

$$\sum_{m=1}^{3} \frac{F(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} \left\{ V^{(2m-1)} e^{kq_m h} + V^{(2m)} e^{-kq_m h} \right\} = 0, \tag{2.19}$$

$$\sum_{m=1}^{3} \frac{F(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} \left\{ V^{(2m-1)} e^{-kq_m h} + V^{(2m)} e^{kq_m h} \right\} = 0, \tag{2.20}$$

$$\sum_{m=1}^{3} \frac{q_m G(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} \left\{ V^{(2m-1)} e^{kq_m h} - V^{(2m)} e^{-kq_m h} \right\} = 0, \tag{2.21}$$

$$\sum_{m=1}^{3} \frac{q_m G(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} \left\{ V^{(2m-1)} e^{-kq_m h} - V^{(2m)} e^{kq_m h} \right\} = 0, \tag{2.22}$$

$$\sum_{m=1}^{3} \frac{H(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} \left\{ V^{(2m-1)} e^{kq_m h} + V^{(2m)} e^{-kq_m h} \right\} = 0, \tag{2.23}$$

$$\sum_{m=1}^{3} \frac{H(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} \left\{ V^{(2m-1)} e^{-kq_m h} + V^{(2m)} e^{kq_m h} \right\} = 0. \tag{2.24}$$

The symmetry of the single layer under consideration about $x_2 = 0$ allows us to simplify the system of 6 homogeneous equations in 6 unknowns (2.19)–(2.24) into two systems of 3 equations in 3 unknowns, which then yield the dispersion relations associated with flexural (symmetric) and extensional (anti-symmetric) waves. The full dispersion relation from the layer is formed from the product of these two relations.

Anti-symmetric solutions (in V) are obtained by adding equation (2.19) to (2.20), (2.23) to (2.24), and subtracting (2.21) from (2.22), to obtain

$$\begin{aligned} \sum_{m=1}^{3} \frac{F(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} V_m^+ C_m &= 0, \\ \sum_{m=1}^{3} \frac{q_m G(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} V_m^+ S_m &= 0, \\ \sum_{m=1}^{3} \frac{F(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} V_m^+ C_m &= 0, \end{aligned} \tag{2.25}$$

within which $C_m = \cosh(kq_m h)$, $S_m = \sinh(kq_m h)$ and $V_m^+ = V^{(2m-1)} + V^{(2m)}$ $(m = 1, 2, 3)$. Similarly, the system of equations associated with symmetric solutions are obtained by subtracting equations (2.19) from (2.20), (2.23) from (2.24), and adding (2.21) to (2.22), to yield

$$\begin{aligned} \sum_{m=1}^{3} \frac{F(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} V_m^- S_m &= 0, \\ \sum_{m=1}^{3} \frac{q_m G(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} V_m^- C_m &= 0, \\ \sum_{m=1}^{3} \frac{F(q_m, \rho v^2)}{\zeta(q_m, \rho v^2)} V_m^- S_m &= 0, \end{aligned} \tag{2.26}$$

where $V_m^- = V^{(2m-1)} - V^{(2m)}$. The condition that the system of equations (2.25) and (2.26) admit a non-trivial solution gives rise to the dispersion relations associated with flexural and extensional waves, respectively.

2.1 Flexural waves

For flexural waves equation (2.26) is satisfied by taking $V^{(2m-1)} = V^{(2m)}$, thus U, V and W have the form

$$\begin{gathered}U = 2\sin\theta \sum_{m=1}^{3} iq_m \frac{f(q_m,\rho v^2)}{\zeta(q_m,\rho v^2)} S_m V^{(2m)}, \quad V = 2\sum_{m=1}^{3} C_m V^{(2m)}, \\ W = 2\cos\theta \sum_{m=1}^{3} iq_m \frac{h(q_m,\rho v^2)}{\zeta(q_m,\rho v^2)} S_m V^{(2m)},\end{gathered} \tag{2.27}$$

and equation (2.25) possesses non-trivial solutions provided

$$\Upsilon \begin{vmatrix} F(q_1,\rho v^2)C_1 & F(q_2,\rho v^2)C_2 & F(q_3,\rho v^2)C_3 \\ q_1G(q_1,\rho v^2)S_1 & q_2G(q_2,\rho v^2)S_2 & q_3G(q_3,\rho v^2)S_3 \\ H(q_1,\rho v^2)C_1 & H(q_2,\rho v^2)C_2 & H(q_3,\rho v^2)C_3 \end{vmatrix} = 0, \tag{2.28}$$

where $\Upsilon = \prod_{m=1}^{3} \zeta(q_m,\rho v^2)^{-1}$.

Expanding the determinant in equation (2.28) provides an explicit representation of the dispersion relation associated with flexural waves, which after a little algebraic manipulation, may be cast in the form

$$\begin{aligned}q_1G(q_1,\rho v^2)\psi(q_2,q_3,\rho v^2)(q_2^2-q_3^2)T_1 - q_2G(q_2,\rho v^2)\psi(q_1,q_3,\rho v^2)(q_1^2-q_3^2)T_2 \\ + q_3G(q_3,\rho v^2)\psi(q_1,q_2,\rho v^2)(q_1^2-q_2^2)T_3 = 0,\end{aligned} \tag{2.29}$$

within which $T_m = \tanh(kq_mh)$ $(m=1,2,3)$ and $\psi(a,b,\rho v^2)$ is defined as

$$\psi(a,b,\rho v^2) = \gamma_{21}\gamma_{23}\psi^{(1)}a^2b^2 + \gamma_{21}\gamma_{23}(\gamma_{23}-\gamma_{21})(\rho v^2-\mu_5)(a^2+b^2) + \psi^{(2)},$$

where

$$\begin{aligned}\psi^{(1)} &= (\gamma_{21}\cos^2\theta + \gamma_{23}\sin^2\theta + \rho v^2)(\gamma_{23}-\gamma_{21}) - \gamma_{21}\{\delta\cos^2\theta - \gamma_{13}\sin^2\theta\} \\ &\quad + \gamma_{23}\{\epsilon\sin^2\theta - \gamma_{31}\cos^2\theta\}, \\ \psi^{(2)} &= (\rho v^2-\mu_5)\Big\{\gamma_{21}(\gamma_{23}-\sigma_2)\{\delta\cos^2\theta - \gamma_{13}\sin^2\theta + \rho v^2\} \\ &\qquad - \gamma_{23}(\gamma_{21}-\sigma_2)\{\epsilon\sin^2\theta - \gamma_{13}\sin^2\theta + \rho v^2\}\Big\}.\end{aligned}$$

2.2 Extensional waves

The dispersion relation associated with extensional waves is obtained by taking $V^{(2m-1)} = -V^{(2m)}$ and from equation (2.26) we therefore require that

$$\Upsilon \begin{vmatrix} F(q_1,\rho v^2)S_1 & F(q_2,\rho v^2)S_2 & F(q_3,\rho v^2)S_3 \\ q_1G(q_1,\rho v^2)C_1 & q_2G(q_2,\rho v^2)C_2 & q_3G(q_3,\rho v^2)C_3 \\ H(q_1,\rho v^2)S_1 & H(q_2,\rho v^2)S_2 & H(q_3,\rho v^2)S_3 \end{vmatrix} = 0. \tag{2.30}$$

The associated forms of U, V and W are now

$$U = 2\sin\theta \sum_{m=1}^{3} iq_m \frac{f(q_m,\rho v^2)}{\zeta(q_m,\rho v^2)} S_m V^{(2m)}, \quad V = 2\sum_{m=1}^{3} C_m V^{(2m)}, \\ W = 2\cos\theta \sum_{m=1}^{3} iq_m \frac{h(q_m,\rho v^2)}{\zeta(q_m,\rho v^2)} S_m V^{(2m)}. \tag{2.31}$$

Evaluating the determinant in equation (2.30) leads to the dispersion relation associated with extensional waves, an explicit representation of which is given, after a little algebraic manipulation, by

$$q_1G(q_1,\rho v^2)\psi(q_2,q_3,\rho v^2)(q_2^2-q_3^2)T_2T_3 - q_2G(q_2,\rho v^2)\psi(q_1,q_3,\rho v^2)(q_1^2-q_3^2)T_1T_3 \\ + q_3G(q_3,\rho v^2)\psi(q_1,q_2,\rho v^2)(q_1^2-q_2^2)T_1T_2 = 0, \tag{2.32}$$

in which $G(q,\rho v^2)$ and $\psi(a,b,\rho v^2)$ are defined immediately below equations (2.18) and (2.29), respectively.

3 Numerical results

Various numerical solutions of the dispersion relations (2.29) and (2.32) are presented here for a set of parameters, at various values of θ, generated using the Mooney-Rivlin strain energy function

$$W = \frac{\mathcal{C}_1}{2}(\lambda_1^2+\lambda_2^2+\lambda_3^2-3) + \frac{\mathcal{C}_2}{2}(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2-3), \tag{3.1}$$

in which $\mathcal{C}_1$ and $\mathcal{C}_2$ are material constants and λ_i is the principal stretch along Ox_i. From equation (3.1) the corresponding values of γ_{ij} and β_{ij} are obtained in the form

$$\gamma_{ij} = \lambda_i^2(\mathcal{C}_1 + 2\mathcal{C}_2\lambda_k^2), \qquad 2\beta_{ij} = (\lambda_i^2+\lambda_j^2)(\mathcal{C}_1+2\mathcal{C}_2\lambda_k^2), \tag{3.2}$$

for $i,j,k \in \{1,2,3 : k \neq i, k \neq j\}$. For all numerical cases discussed in this paper in the high wave number region one value of q is imaginary, with the others either real or forming a complex conjugate pair. Moreover, if without loss of generality we assume $q_1 = i\hat{q}_1$ then $\hat{q}_1 \to 0$ as $kh \to \infty$. A trivial examination of equation (2.14) shows that there are two possible wave speeds meeting this

Mooney-Rivlin Material $\mathcal{C}_1 = 2.0$, $\mathcal{C}_2 = 0.4$, $\lambda_1^2 = 3.695$, $\lambda_2^2 = 0.7$, $\lambda_3^2 = 0.387$, $\sigma_2 = 0.8$			
$\gamma_{12} = 8.533$	$\gamma_{21} = 1.617$		$2\beta_{12} = 10.149$
$\gamma_{13} = 9.459$	$\gamma_{31} = 0.990$		$2\beta_{13} = 10.449$
$\gamma_{23} = 3.469$	$\gamma_{32} = 1.916$		$2\beta_{23} = 5.385$
Variation of the limiting speed and v_R with θ			
θ	$\sqrt{\mu_4/\rho}$	$\sqrt{\mu_5/\rho}$	v_R
15	1.536	1.248	1.232
45	2.286	2.286	2.179
75	2.844	2.982	2.836

Table 1: Material parameters used in generating the figures.

criteria, these being $v = \sqrt{\mu_4/\rho}$ and $v = \sqrt{\mu_5/\rho}$. The possible limiting wave speeds exemplified by $|q_1| \to |q_2|$ as $kh \to \infty$, see e.g. Rogerson [8], are left for further work. At this stage we note that throughout this paper when we refer to phase speed we strictly mean a scaled phase speed defined by $\bar{v}^2 = \rho v^2$. A similar notion will also apply to the Raleigh surface wave speed.

The set of figures (1)–(6) have been generated for the material parameters shown in table (1). For these parameters the high wave limit of the harmonics changes from $\rho v^2 \to \mu_5$ to $\rho v^2 \to \mu_4$ as the angle of propagation increases, with the two shear wave speeds being equal when $\theta \approx 45^o$. Figures (1) and (2) show the first twenty-five branches of the flexural and extensional dispersion relations, respectively, for an angle of propagation $\theta = 15^o$. The high wave number limit of the fundamental mode is distinct from that of the harmonics, with the fundamental mode tending (from below) to surface wave with speed $\bar{v}_R = 1.232$ and the harmonics asymptoting (from above) to a shear wave speed, given by $\rho v^2 = \mu_5$. It is reiterated that this high wave number limit occurs when one of q_1, q_2 and q_3 is imaginary, the other two being real or (as in this case) complex conjugates. The wave speed region where this holds true lies within $\mu_5 \leqslant \rho v^2 \leqslant \mu_4$. In the low wave number region only the fundamental mode has a finite wave speed in the flexural case, whilst the extensional dispersion relation possess a finite wave speed limit for both the fundamental mode and first harmonic. We observe a 'ghost' line forming around $v \approx 1.536$, which is formed by the higher harmonics flattening and coming close together as they pass through the value of the second shear wave, given by $\rho v^2 = \mu_4$, associated with this layer.

[Figures 1 and 2 about here.]

When the angle of propagation is increased to 45^o in figures (3) and (4) we find that for these parameters the two shear waves associated with $q = 0$ have approximately the same speed (identical to eight decimal places), this being $\bar{v} = 2.286$. The high wave number phase speed limit of the harmonics is therefore now $\rho v^2 = \mu_4 = \mu_5 = 5.224$. In general, the angle at which this occurs, if any, is readily deduced from the definitions of μ_4 and μ_5 and will arise if

$$\tan^2\theta = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_1^2(1 - \lambda_1^2\lambda_2^4)} \geqslant 0.$$

Additionally, the most striking change to the dispersion curves is that the fundamental mode now has a high wave number limit quite distinct from that of the harmonics. The fundamental mode has a limiting wave speed of $\bar{v} = 2.179$, which corresponds to a Rayleigh surface wave. As the two shear wave speeds associated with $q = 0$ are almost equal figures (3) and (4) do not exhibit the ghost lines seen in previous figures, although the harmonics still give the impression of crossing over.

[Figures 3 and 4 about here.]

When the angle of propagation is increased to 75^o, figures (5) and (6), we see that now $\mu_4 < \mu_5$ and the high wave number limit of the harmonics is given by $\rho v^2 = \mu_4$. The fundamental mode of both extensional and flexural waves tends to a surface wave speed of $\bar{v}_R = 2.8359$. Figure (6) shows most clearly the two possible limiting values of extensional waves as $kh \to 0$. These two limiting wave speeds, along with the surface wave equation, will be discussed in more detail in a later section.

[Figures 5 and 6 about here.]

From these numerical solutions we have three low wave number limiting wave speeds associated with the single layer, which is in direct contrast to the two possible limiting wave speeds found by Rogerson and Fu [7] for a direction of propagation parallel to the Ox_1 axis and under an assumption of plane strain. However, three low wave number limiting wave speeds have previously been observed by Nayfeh [14] when presenting numerical solutions for wave propagation at an arbitrary angle in a multi-layered linear anisotropic media. The three limiting wave speeds obtained by Nayfeh are characterised as one quasi-longitudinal and two quasi-shear. Similar behaviour was also obtained by Kulkarni and Pagano [15] who considered vibrations in fibre reinforced laminates.

A further feature of the graphs shown here for figures (1)–(4) not seen in the corresponding dispersion curves of a single layer plate under plane strain is that pairs of harmonics appear to intersect. This is most marked in figures (1) and (2), for $\theta = 15^o$, in which it appears to the eye that there are two sets of harmonics intersecting each other. We shall now consider this interesting behaviour further by considering the dispersion relation in the special case $\theta = \pi/2$.

$\theta = 0$ or $\pi/2$

For some of the dispersion curves generated within this section the harmonics appear to intersect and cross. This numerical behaviour, especially evident in figures (1) and (2), is not present in the dispersion curves obtained for a single layer by Rogerson and Fu [7] and Rogerson [8] under the assumption of plane strain and with a direction of propagation parallel to one the principal directions of the right Cauchy-Green strain tensor. To investigate this further we shall consider the special case $\theta = 0$ and $\theta = \pi/2$, which correspond to a direction of wave propagation parallel to one of the in-plane principal directions. Setting $\theta = \pi/2$ in equations (2.10)–(2.13) gives

$$\begin{gathered} U\{q^2 B_{2121} + B_{1122} - B_{1111} + \rho v^2\} - iP = 0, \\ iU\{q^2(B_{1221} + B_{2211} - B_{2222}) + B_{1212} - \rho v^2\} - q^2 P = 0, \\ W\{q^2 B_{2323} - B_{1313} + \rho v^2\} = 0, \end{gathered} \tag{3.3}$$

and the incompressibility constraint now takes the simplified form $iU + qV = 0$. Equations $(3.3)_{1,2}$ are identical to those previously obtained by Ogden and Roxburgh [6] derived under an assumption of plane strain, whilst equation $(3.3)_3$ uncouples from the other two equations due to its sole dependence on W, implying that either $W = 0$ or

$$q^2 = \frac{\gamma_{13} - \rho v^2}{\gamma_{23}}. \tag{3.4}$$

The secular equation (2.14) similarly simplifies when $\theta = \pi/2$, and may be expressed as

$$\{\gamma_{23} q^2 - \gamma_{13} + \rho v^2\}\{\gamma_{21} q^4 + (\rho v^2 - 2\beta_{12}) q^2 + \gamma_{12} - \rho v^2\} = 0,$$

from which we can say without loss of generality that q_1^2 and q_2^2 satisfy

$$\gamma_{21} q^4 + (\rho v^2 - 2\beta_{12}) q^2 + \gamma_{12} - \rho v^2 = 0, \tag{3.5}$$

and

$$q_3^2 = \frac{\gamma_{13} - \rho v^2}{\gamma_{23}}, \tag{3.6}$$

where we note in the notation of Ogden and Roxburgh [6] that $\gamma_{12} = \alpha$, $\beta_{12} = \beta$ and $\gamma_{21} = \gamma$, equation (3.5) then being identical to their secular equation. Equation (3.6), together with (3.4), imply that W is dependent only on q_3, whilst U and V are dependent only on q_1 and q_2. The solutions of the eigenfunctions U, V, and W are therefore

$$U = \sum_{m=1}^{2} i q_m \left(V^{(2m-1)} E_m^+ - V^{(2m)} E_m^- \right), \qquad V = \sum_{m=1}^{2} \left(V^{(2m-1)} E_m^+ + V^{(2m)} E_m^- \right),$$
$$W = W^{(5)} E_3^+ + W^{(6)} E_3^-. \tag{3.7}$$

On inserting the form of solutions represented by equation (3.7) into the equations of incremental traction (2.6)–(2.8) we obtain

$$\begin{aligned}\frac{\tau_1}{ik} &= \sum_{i=1}^{2} f(q_m)\left\{V^{(2m-1)}E_m^+ + V^{(2m)}E_m^-\right\},\\ \frac{\tau_2}{k} &= \sum_{i=1}^{2} q_m F(q_m)\left\{V^{(2m-1)}E_m^+ - V^{(2m)}E_m^-\right\},\\ \frac{\tau_3}{k} &= q_3\left\{W^{(5)}E_3^+ - W^{(6)}E_3^-\right\},\end{aligned} \tag{3.8}$$

where $f(q_m) = \gamma_{21}(q_m^2+1) - \sigma_2$ and $F(q)$ is defined as $F(q_i) = f(q_j)$ for $i, j \in \{1, 2 : i \neq j\}$. In obtaining equation $(3.8)_2$ use has also been made of the fact $\gamma_{21}(q_1^2 + q_2^2) = \alpha - \rho v^2$, which may be deduced from equation (3.5).

The dispersion relation associated with an angle of propagation $\theta = \pi/2$ is obtained from equation (3.8) by imposing our usual traction free boundary conditions on the upper and lower surfaces, namely $\tau_i = 0$ for $i = 1, 2, 3$ on $x_2 = \pm h$. Applying the appropriate boundary conditions to the third equation of incremental traction gives

$$\begin{aligned} q_3\left(W^{(5)}e^{kq_3h} - W^{(6)}e^{-kq_3h}\right) &= 0,\\ q_3\left(W^{(5)}e^{-kq_3h} - W^{(6)}e^{kq_3h}\right) &= 0,\end{aligned}$$

which is satisfied non-trivially if $q_3 = 0$ or $\sinh 2kq_3h = 0$. The solution $q_3 = 0$, in conjunction with equation (3.6), gives rise to a non-dispersive shear wave propagating with speed $v = \sqrt{\gamma_{13}/\rho}$ along the Ox_1 axis, whilst $\sinh 2kq_3h = 0$, with $q_3 \neq 0$, implies that q_3 is imaginary. The non-dispersive shear wave associated with $q_3 = 0$ is termed an exceptional wave. These are homogeneous plane waves propagating such that the traction on a family of parallel planes is unaffected, see Chadwick and Whitworth [16]. Setting $q_3 = i\hat{q}_3$ leads to solutions for W and τ_3 in the form

$$\begin{aligned} W &= (W^{(5)} + W^{(6)})\cos k\hat{q}_3x_2 + i(W^{(5)} - W^{(6)})\sin k\hat{q}_3x_2,\\ \frac{\tau_3}{ik} &= \hat{q}_3\left\{(W^{(5)} - W^{(6)})\cos k\hat{q}_3x_2 + i(W^{(5)} + W^{(6)})\sin k\hat{q}_3x_2\right\},\end{aligned}$$

with the subsequent boundary conditions associated with $\tau_3 = 0$ on $x_2 = \pm h$ being

$$\hat{q}_3\left\{(W^{(5)} - W^{(6)})\cos k\hat{q}_3h + i(W^{(5)} + W^{(6)})\sin k\hat{q}_3h\right\} = 0, \tag{3.9}$$

$$\hat{q}_3\left\{(W^{(5)} - W^{(6)})\cos k\hat{q}_3h - i(W^{(5)} + W^{(6)})\sin k\hat{q}_3h\right\} = 0. \tag{3.10}$$

This system of equations may be manipulated into extensional and flexural waves by appropriate addition/subtraction of the boundary conditions, thus

$$\hat{q}_3(W^{(5)} - W^{(6)})\cos k\hat{q}_3h = 0, \quad \text{and} \quad \hat{q}_3(W^{(5)} + W^{(6)})\sin k\hat{q}_3h = 0. \tag{3.11}$$

The condition that $(3.11)_1$ and $(3.11)_2$ have non-trivial solutions gives rise to the dispersion relations associated with flexural and extensional waves, respectively, from which exact solutions of the phase speed can be obtained. Noting that $q_3 \neq 0$, we have for flexural waves $\cos k\hat{q}_3 h = 0$ implying that

$$\hat{q}_3 = \left(n + \frac{1}{2}\right)\frac{\pi}{kh},$$

which, with equation (3.6), gives the explicit representation of the phase speed

$$\rho v^2 = \gamma_{13} + \gamma_{23}\left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{(kh)^2}. \tag{3.12}$$

Similarly, for extensional waves $\sin k\hat{q}_3 h = 0$, implying that

$$\hat{q}_3 = \frac{n\pi}{kh},$$

which, in conjunction with equation (3.6), gives the phase speed as

$$\rho v^2 = \gamma_{13} + \gamma_{23}\left(\frac{n\pi}{kh}\right)^2, \tag{3.13}$$

equations (3.12) and (3.13) relating to horizontally polarised shear waves.

If we now apply the boundary conditions to the two equations for τ_1 and τ_2 the resulting system of equations may then be separated into symmetric and anti-symmetric solutions by appropriate manipulations, which yield non-trivial solutions provided either

$$\begin{vmatrix} f(q_1)S_1 & f(q_2)S_2 \\ q_1 f(q_2)C_1 & q_2 f(q_1)C_2 \end{vmatrix} = 0, \text{ or } \quad \begin{vmatrix} f(q_1)C_1 & f(q_2)C_2 \\ q_1 f(q_2)S_1 & q_2 f(q_1)S_2 \end{vmatrix} = 0. \tag{3.14}$$

Expanding the two determinants in equation (3.14) gives the dispersion relations associated respectively with extensional and flexural waves, namely

$$\begin{aligned} q_1 f(q_2)^2 C_1 S_2 - q_2 f(q_1)^2 S_1 C_2 = 0, \\ q_1 f(q_2)^2 S_1 C_2 - q_2 f(q_1)^2 C_1 S_2 = 0. \end{aligned} \tag{3.15}$$

A similar analysis in the case $\theta = 0$ yields the analogous forms of (3.12) and (3.13) as

$$\rho v^2 = \gamma_{31} + \gamma_{21}\left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{(kh)^2} \tag{3.16}$$

and

$$\rho v^2 = \gamma_{31} + \gamma_{21}\left(\frac{n\pi}{kh}\right)^2. \tag{3.17}$$

The appropriate forms of equations (3.15) are also given by

$$\begin{aligned} q_1 \hat{f}(q_2)^2 C_1 S_2 - q_2 \hat{f}(q_1)^2 S_1 C_2 = 0, \\ q_1 \hat{f}(q_2)^2 S_1 C_2 - q_2 \hat{f}(q_1)^2 C_1 S_2 = 0, \end{aligned} \tag{3.18}$$

in which $\hat{f}(q) = \gamma_{23}(q^2+1) - \sigma_2$.

It would seem that we must consider the dispersion relations $(3.15)_1$ and (3.13) or $(3.15)_2$ and (3.12) to adequately compare the dispersion relations for $\theta = \pi/2$ with those previously obtained for $\theta \neq \pi/2$ (and similarly for $\theta = 0$). This is considered in the next two figures, (7) and (8), which show numerical solutions of the dispersion relation for a single layer plate under plane strain $(3.15)_{1,2}$ with the horizontally polarised shear wave (SH) (3.12) and (3.13) for extensional and flexural waves, respectively. The figures have been generated for the material parameters in table (1) and clearly show the two dispersion curves associated with extensional and flexural waves, respectively, intersecting. This explains the apparent intersection of harmonics observed in the numerical section for certain material parameters. The horizontally polarised shear waves presented in equations (3.12) and (3.13) uncouple from the system only in the particular case $\theta = 0$ or $\pi/2$. At other angles of propagation these waves cannot be separated and instead lead to dispersion curves exemplified by figures (1) and (2).

[Figures 7 and 8 about here.]

4 An asymptotic analysis

4.1 Long wavelength limit $kh \to 0$

In the numerical section different low wave number limiting behaviour was observed for extensional and flexural waves. The limiting behaviour of the flexural dispersion relation is similar to that observed for the symmetric 4-ply laminate and for a single layer, under a plane strain approximation, in which only the fundamental mode retains a finite value as $kh \to 0$, see Rogerson and Sandiford [9] and Rogerson and Fu [7], respectively. For extensional waves however, both the fundamental mode and first harmonic have finite wave speed in this limit, thus giving a total of three long wave limits for the single layer. This behaviour is in direct contrast to that arising for a single layer plate when the angle of propagation is along a principal direction, in which only one limit is seen for both extensional and flexural waves. The low wave number behaviour is now investigated further by considering analytical representations of the limiting wave speeds.

Before we derive analytical expressions for either the extensional or flexural low wave number limiting wave speeds it will be of aid to first rewrite the functions $G(q, \rho v^2)$ and $\psi(a, b, \rho v^2)$ defined below equations (2.18) and (2.29), respectively, in the form

$$\begin{aligned} G(q,\rho v^2) &= -\bar{v}^4 + \bar{v}^2\big(G_0^{(2)} + G_2^{(2)}q^2\big) + G_0^{(0)} + G_2^{(0)}q^2 + G_4^{(0)}q^4, \\ \psi(a,b,\rho v^2) &= \psi_0^{(4)}\bar{v}^4 + \bar{v}^2\left(\psi_{ab}^{(2)}a^2b^2 + \psi_{a+b}^{(2)}(a^2+b^2) + \psi_0^{(2)}\right) \\ &\quad + \psi_{ab}^{(0)}a^2b^2 + \psi_{a+b}^{(0)}(a^2+b^2) + \psi_0^{(0)}, \end{aligned} \tag{4.1}$$

where $G_n^{(m)}$ denotes the O(1) coefficient of $\bar{v}^m q^n$, with $\psi_{ab}^{(m)}$, $\psi_{a+b}^{(m)}$ and $\psi_0^{(m)}$ denoting respectively the coefficients of $a^2b^2\bar{v}^m$, $(a^2+b^2)\bar{v}^m$ and $\bar{v}^m$, these quantities given explicitly by

$$\begin{aligned}
G_0^{(0)} =& \sin^2\theta\cos^2\theta\big\{\gamma_{21}\delta + \gamma_{23}\epsilon + \delta^2 - \gamma_{13}\gamma_{31} - 4\beta_{13}\beta_{23} + 2\beta_{13}\sigma_2\big\} \\
& - \gamma_{13}\sin^4\theta\big(2\beta_{12} + \gamma_{21} - \sigma_2\big) - \gamma_{31}\cos^4\theta\big(2\beta_{23} + \gamma_{23} - \sigma_2\big), \\
G_2^{(0)} =& \gamma_{23}\big(\gamma_{31}\cos^2\theta + \sin^2\theta(2\beta_{12} + \gamma_{21} - \sigma_2)\big) \\
& + \gamma_{21}\big(\gamma_{13}\sin^2\theta + \cos^2\theta(2\beta_{23} + \gamma_{23} - \sigma_2)\big), \\
G_4^{(0)} =& -\gamma_{21}\gamma_{23}, \qquad G_2^{(2)} = -(\gamma_{21} + \gamma_{23}), \\
G_0^{(2)} =& \big(\gamma_{13} + \gamma_{21} + 2\beta_{12} - \sigma_2\big)\sin^2\theta + \big(\gamma_{31} + \gamma_{23} + 2\beta_{23} - \sigma_2\big)\cos^2\theta, \\
\psi_0^{(0)} =& \mu_5\left\{\gamma_{23}(\gamma_{21} - \sigma_2)(\epsilon\sin^2\theta - \gamma_{31}\cos^2\theta) - \gamma_{21}(\gamma_{23} - \sigma_2)(\delta\cos^2\theta + \gamma_{13}\sin^2\theta)\right\}, \\
\psi_{a+b}^{(0)} =& \gamma_{21}\gamma_{23}\mu_5(\gamma_{21} - \gamma_{23}), \\
\psi_{ab}^{(0)} =& \gamma_{21}\gamma_{23}\left\{\sin^2\theta(\gamma_{23}(\gamma_{23} - \gamma_{21}) + \gamma_{23}\epsilon - \gamma_{13}\gamma_{21})\right. \\
& \left. + \cos^2\theta(\gamma_{21}(\gamma_{23} - \gamma_{21}) - \gamma_{23}\gamma_{31} - \gamma_{21}\delta)\right\}, \\
\psi_0^{(2)} =& \gamma_{21}(\gamma_{23} - \sigma_2)(\delta\cos^2\theta - \gamma_{13}\sin^2\theta) - \gamma_{23}(\gamma_{21} - \sigma_2)(\epsilon\sin^2\theta - \gamma_{31}\cos^2\theta) \\
& + \sigma_2\mu_5(\gamma_{21} - \gamma_{23}), \\
\psi_{a+b}^{(2)} =& \psi_{ab}^{(2)} = \gamma_{21}\gamma_{23}(\gamma_{23} - \gamma_{21}), \qquad \psi_0^{(4)} = (\gamma_{23} - \gamma_{21})\sigma_2.
\end{aligned}$$

We shall now consider in turn the limiting behaviour of extensional and flexural waves in the low wave number regime.

Flexural waves

The long wave limiting form of the flexural dispersion relation is obtained by allowing $kh \to 0$ in equation (2.29), whilst assuming the speed of wave propagation remains finite, which gives to leading order

$$\begin{aligned}
kh\left\{q_1^2G(q_1,\rho v^2)\psi(q_2,q_3,\rho v^2)(q_2^2 - q_3^2) - q_2^2G(q_2,\rho v^2)\psi(q_1,q_3,\rho v^2)(q_1^2 - q_3^2)\right. \\
\left. + q_3^2G(q_3,\rho v^2)\psi(q_1,q_2,\rho v^2)(q_1^2 - q_2^2)\right\} = 0. \quad (4.2)
\end{aligned}$$

On inserting the representations of $G(q,\rho v^2)$ and $\psi(a,b,\rho v^2)$ shown in equations (4.1), making use of equation (2.14), and after a little algebraic manipulation, we find that either $q_1^2 = q_2^2$, $q_1^2 = q_3^2$,

$q_2^2 = q_3^2$ or

$$\bar{v}^6 \left\{ \psi_{a+b}^{(2)} + G_2^{(0)} \psi_0^{(4)} - (\gamma_{21} + \gamma_{23}) \psi_0^{(4)} + \psi_{ab}^{(2)} + \psi_{a+b}^{(2)} \right\}$$
$$+ \bar{v}^4 \left\{ \psi_{a+b}^{(0)} - G_0^{(2)} + G_2^{(2)} \psi_0^{(2)} + G_2^{(0)} \psi_0^{(4)} + \mu_1 \psi_0^{(4)} \right.$$
$$\left. -(\gamma_{21} + \gamma_{23}) \psi_0^{(2)} + -\mu_2 \psi_{a+b}^{(2)} + \psi_{a+b}^{(0)} - (\mu_4 + \mu_5) \psi_{ab}^{(2)}) + \psi_{ab}^{(0)} \right\}$$
$$+ \bar{v}^2 \left\{ -G_0^{(2)} \psi_{a+b}^{(0)} + G_2^{(2)} \psi_0^{(0)} - G_0^{(0)} \psi_{a+b}^{(2)} + G_0^{(2)} \psi_0^{(2)} + \mu_1 \psi_0^{(2)} \right.$$
$$\left. -(\gamma_{21} + \gamma_{23}) \psi_0^{(0)} + \mu_3 \psi_{ab}^{(2)} - \mu_2 \psi_{a+b}^{(0)} + \mu_4 \mu_5 \psi_{ab}^{(2)} - (\mu_4 + \mu_5) \psi_{ab}^{(0)} \right\}$$
$$- G_0^{(0)} \psi_{a+b}^{(0)} + G_2^{(0)} \psi_4^{(0)} + \mu_1 \psi_4^{(0)} + \mu_3 \psi_{a+b}^{(0)} + \mu_4 \mu_5 \psi_{ab}^{(0)} = 0. \quad (4.3)$$

Equation (4.3) is a cubic in $\bar{v}^2$ and suggests that there are in fact three finite limiting wave speeds as $kh \to 0$. However, numerical investigation of equation (4.3) shows that two of the solutions are spurious and arise from $\rho v^2 = \mu_5$ and $q_1 = q_2$. These two solutions are spurious in the sense that they lead to non-dispersive solutions. The remaining solution of equation (4.3) is then the real limiting wave speed for fundamental mode as $kh \to 0$, and agrees with the numerical solutions of the dispersion relation.

Extensional waves

The leading order form of the extensional dispersion relation (2.32) for small kh is

$$q_1 q_2 q_3 kh \left\{ G(q_1, \rho v^2) \psi(q_2, q_3, \rho v^2)(q_2^2 - q_3^2) - G(q_2, \rho v^2) \psi(q_1, q_3, \rho v^2)(q_1^2 - q_3^2) \right.$$
$$\left. + G(q_3, \rho v^2) \psi(q_1, q_2, \rho v^2)(q_1^2 - q_2^2) \right\} = 0. \quad (4.4)$$

The common factors $q_1 q_2 q_3 = 0$ lead to spurious roots in the sense of non-dispersive shear wave speeds $\rho v^2 = \mu_4$ and $\rho v^2 = \mu_5$. If we now insert the representations of $G(q_m, \rho v^2)$ and $\psi(a, b, \rho v^2)$ shown in equations (4.1) into the above equation, and remove the common factor $q_1 q_2 q_3$, we find either

$$(q_1^2 - q_2^2)(q_1^2 - q_3^2)(q_2^2 - q_3^2) = 0,$$

or

$$- \psi_{ab}^{(2)} \bar{v}^6 + \bar{v}^4 \left(-\psi_{ab}^{(0)} + G_0^{(2)} \psi_{ab}^{(2)} + G_4^{(0)} \psi_0^{(4)} - G_2^{(2)} \psi_{a+b}^{(2)} \right)$$
$$+ \bar{v}^2 \left(G_0^{(2)} \psi_{ab}^{(0)} - G_2^{(2)} \psi_{a+b}^{(0)} + G_4^{(0)} \psi_0^{(2)} - G_2^{(0)} \psi_{a+b}^{(2)} + G_0^{(0)} \psi_{ab}^{(2)} \right)$$
$$+ \left(G_0^{(0)} \psi_{ab}^{(0)} - G_2^{(0)} \psi_{a+b}^{(0)} + G_4^{(0)} \psi_0^{(0)} \right) = 0. \quad (4.5)$$

Equation (4.5) is again a cubic in $\bar{v}^2$, suggesting that we again have three finite limiting wave speeds as $kh \to 0$. However, in this case numerical solutions of this equation shows that *one* of these roots is spurious and is a non-dispersive shear wave speed associated with $q_1 = q_2$. The remaining two roots are then the two limiting wave speeds of the extensional wave dispersion relation and agree with results obtained numerically.

Numerical solutions of (4.5) shows that as $\theta \to 90^o$ one of the solutions is $\rho v^2 = \gamma_{13}$, the speed of the non-dispersive (exceptional) polarised (SH) shear wave in the case $\theta = 90^o$, associated with $n = 0$ in equation (3.13). A similar situation occurs as $\theta \to 0^o$, when one of the solutions of equation (4.5) is $\rho v^2 \to \gamma_{31}$, which is again a non-dispersive polarised (SH) shear wave. This is relatively easy to show by setting $\theta = 0^o$ or $\theta = 90^o$ in equation (4.5), to obtain

$$(\gamma_{31} - \rho v^2)(\rho v^2 - 2\beta_{23} - 2\gamma_{23} + 2\sigma_2)\big(\rho v^2(\gamma_{23} - \gamma_{21}) - \gamma_{21}\delta - \gamma_{23}\gamma_{31}\big) = 0, \tag{4.6}$$

or

$$(\rho v^2 - \gamma_{13})(\rho v^2 - 2\beta_{12} - 2\gamma_{21} + 2\sigma_2)\big(\rho v^2(\gamma_{21} - \gamma_{23}) - \gamma_{23}\epsilon - \gamma_{21}\gamma_{13}\big) = 0. \tag{4.7}$$

The first factor of equation (4.7) is the exceptional (SH) wave associated with a direction of propagation along the Ox_1 axis, the second factor is the appropriate long wave limit obtained by Rogerson [8], whilst the third factor is a spurious solution associated with $q_1 = q_3$. The three factors of equation (4.6) are the analogous factors associated with a direction of propagation along Ox_3.

4.2 Surface waves

We first consider the high wave limiting behaviour of the extensional and flexural dispersion relations when q_1, q_2 and q_3 are either all purely real or one real with one complex conjugate pair. From previous investigations, see for example Ogden and Roxburgh [6], we expect this situation to give rise to a surface wave in the high wave number limit, and this is supported numerically. Allowing $kh \to \infty$ in either equation (2.29) or (2.32) then gives

$$\begin{aligned} q_1 G(q_1, \rho v^2)\psi(q_2, q_3, \rho v^2)(q_2^2 - q_3^2) - q_2 G(q_2, \rho v^2)\psi(q_1, q_3, \rho v^2)(q_1^2 - q_3^2) \\ + q_3 G(q_3, \rho v^2)\psi(q_1, q_2, \rho v^2)(q_1^2 - q_2^2) = 0. \end{aligned} \tag{4.8}$$

On inserting the definitions of $G(q)$ and $\psi(a, b)$ indicated in equation (4.1), and after a little algebraic manipulation, we obtain

$$\begin{aligned} & q_1 q_2 q_3 (q_1 + q_2 + q_3)\{\mathcal{AE} - \mathcal{BF} + \mathcal{CG}\} \\ & + (q_1q_2 + q_1q_3 + q_2q_3)\{\mathcal{AF} + \mathcal{BG} + \mathcal{CE}(q_1^2q_2^2q_3^2) + \mathcal{CF}(q_1^2q_2^2 + q_1^2q_3^2 + q_2^2 + q_3^2) + \mathcal{CG}(q_1^2 + q_2^2 + q_3^2)\} \\ & + (\mathcal{CF} - \mathcal{BE})(q_1^2q_2^2q_3^2) - \mathcal{AF}(q_1^2 + q_2^2 + q_3^2) - \mathcal{AG} + (\mathcal{CG} - \mathcal{BF})(q_1^2q_2^2 + q_1^2q_3^2 + q_2^2q_3^2) = 0, \end{aligned} \tag{4.9}$$

where

$$\mathcal{A} = -\bar{v}^4 + G_0^{(2)}\bar{v}^2 + G_0^{(0)}, \qquad \mathcal{E} = \bar{v}^2\psi_{ab}^{(0)},$$
$$\mathcal{B} = \bar{v}^2 G_2^{(2)} + G_2^{(0)}, \qquad \mathcal{F} = \bar{v}^2\psi_{a+b}^{(2)} + \psi_{a+b}^{(0)},$$
$$\mathcal{C} = G_4^{(0)}, \qquad \mathcal{G} = \bar{v}^4\psi_0^{(4)} + \bar{v}^2\psi_0^{(2)} + \psi_0^{(0)}.$$

The surface wave speed for a particular set of material parameters will clearly depend on both the angle of propagation and σ_2 and this is shown in figures (9) and (10). Both figures have been generated using the material parameters shown in table (1). Figure (9) shows the variation of the surface wave speed with σ_2 for fixed θ. For these material parameters we clearly have a greater range of configurations which will support a surface wave for smaller angles of propagation. For each curve the range of σ_2 for which surface wave propagation is possible is the region bounded by the curve. In addition, the greatest value of v_R for each curve corresponds to a shear wave, the appropriate value of σ_2 then indicating the value at which the surface wave degenerates into a shear wave. Figure (10) shows the variation of the surface wave speed with θ for a fixed value of $\sigma_2 = 0.8$. The solid line indicates the surface wave speed, and the two hashed lines represent the values of the two shear waves. The surface wave speed is clearly bounded by the two shear waves, as it is known that $\rho v_R^2 \leqslant \min(\mu_4, \mu_5)$, with the surface wave degenerating into a shear wave as θ approaches 0^o and 90^o. It is worth noting that this behaviour is particular to these material parameters and is not indicative of the general variation of the surface wave speed with θ. With appropriate choice of σ_2 situations exist in which a surface wave exists for all θ, or conversely does not exist for all θ. We note that the innermost range of σ_2 $(0 \leqslant \theta \leqslant \pi/2)$ indicates the region of stability for the plate in respect of quasi-static surface deformations.

[Figures 9 and 10 about here.]

4.3 Short wavelength limit of the harmonics $kh \to \infty$

Guided by our previous numerical calculations we now seek to obtain an asymptotic representation of the phase speed associated with all harmonics in the high wave number regime. Due to the increased complexity of the dispersion relation associated with a general angle of propagation, when compared with previous plane strain studies, attention is restricted to the two cases $\rho v^2 \to \mu_4$ and $\rho v^2 \to \mu_5$. Numerical calculations indicate that these two limits arise when, without loss of generality, $q_1 = i\hat{q}_1$ is imaginary while q_2 and q_3 are either both purely real or complex conjugates, and that $\hat{q}_1 \to 0$ as $kh \to \infty$. In general q_2 and q_3 retain finite magnitudes. The actual limiting value of the phase speed is dependent on the relative magnitudes of μ_4 and μ_5, with ρv^2 tending to the lower of these two values. An approximation to the phase speed is therefore sought by expanding all terms within the dispersion relation in powers of the small order quantity $\hat{q}_1$. An expansion for the phase speed may be obtained in terms of $\hat{q}_1$ by writing equation (2.14) in the form

$$\rho^2 v^4(\hat{q}_1^2+1)-\rho v^2\{(\gamma_{21}+\gamma_{23})\hat{q}_1^4+\mu_2\hat{q}_1^2+\mu_4+\mu_5\} \\ +\gamma_{21}\gamma_{23}\hat{q}_1^6+\mu_1\hat{q}_1^4+\mu_3\hat{q}_1^2+\mu_4\mu_5=0. \quad (4.10)$$

Equation (4.10) will yield two solutions for ρv^2, corresponding to the two limits $\rho v^2 \to \mu_4$ and $\rho v^2 \to \mu_5$, namely

$$\rho v^2=\mu_4-\hat{q}_1^2\frac{\mu_4(\mu_4-\mu_2)+\mu_3}{\mu_4-\mu_5}+O(\hat{q}_1^4), \quad \text{if } \mu_4<\mu_5, \quad (4.11)$$

$$\rho v^2=\mu_5-\hat{q}_1^2\frac{\mu_5(\mu_5-\mu_2)+\mu_3}{\mu_5-\mu_4}+O(\hat{q}_1^4), \quad \text{if } \mu_5<\mu_4. \quad (4.12)$$

Similar expansions for q_2^2 and q_3^2 are found by eliminating q_3^2 from the standard equations for the sum and the sum of the products of two roots of equation (2.15), thus

$$\gamma_{21}\gamma_{23}q_2^4+q_2^2\{(\gamma_{21}+\gamma_{23})\rho v^2-\gamma_{21}\gamma_{23}\hat{q}_1^2\} \\ +(\rho^2v^4-\mu_2\rho v^2+\mu_3)-\hat{q}_1^2\{(\gamma_{21}+\gamma_{23})\rho v^2-\mu_1\}+O(\hat{q}_1^4)=0, \quad (4.13)$$

which seemingly yields two values of q_2^2. However, it is easily verified that on eliminating q_2^2, rather than q_3^2, a quadratic in q_3^2 is obtained with the same coefficients as equation (4.13). The two roots of this quadratic equation therefore correspond to the values of q_2^2 and q_3^2, and we take without loss of generality q_2^2 to be associated with the positive square root of the discriminant. As equation (4.13) depends on the phase speed, we must insert one of equations (4.11) and (4.12), appropriate to the limiting value of the phase speed, into equation (4.13), before finding the two roots, thus

$$q_2^2=\bar{q}_2^{(0)}+O(\hat{q}_1^2), \qquad q_3^2=\bar{q}_3^{(0)}+O(\hat{q}_1^2), \quad (4.14)$$

where $\bar{q}_2^{(0)}$ and $\bar{q}_3^{(0)}$ are order 1 terms defined as

$$\text{if } \rho v^2\to\mu_4: \quad \bar{q}_2^{(0)},\bar{q}_3^{(0)}=\frac{\mu_1-\mu_4(\gamma_{21}+\gamma_{23})\pm\kappa_1(\mu_4)}{2\gamma_{21}\gamma_{23}},$$

$$\text{if } \rho v^2\to\mu_5: \quad \bar{q}_2^{(0)},\bar{q}_3^{(0)}=\frac{\mu_1-\mu_5(\gamma_{21}+\gamma_{23})\pm\kappa_1(\mu_5)}{2\gamma_{21}\gamma_{23}},$$

within which

$$\kappa_1(\rho v^2)=\Big\{\big[(\gamma_{21}+\gamma_{23})\rho v^2-\mu_1\big]^2-4\gamma_{21}\gamma_{23}(\rho^2v^4-\mu_2\rho v^2+\mu_3)\Big\}^{1/2}.$$

We shall now consider extensional and flexural waves separately.

4.3.1 Flexural waves

When deriving the leading order expansions of the dispersion relation care must be taken as the order of $\psi(a,b,\rho v^2)$ changes depending on the limiting value of ρv^2. In the limit $\rho v^2\to\mu_5$ a glance at the definition of $\psi(a, b, \rho v^2)$, see immediately after equation (2.29), shows that the leading order term (which is in general O(1)) will vanish, thus changing the order of $\psi(a, b, \rho v^2)$ and subsequently the leading order term of the dispersion relation. The two limits $\rho v^2 \to \mu_4$ and $\rho v^2 \to \mu_5$ must therefore be considered separately.

(i) Limit $\rho v^2 \to \mu_4$

In the case $\rho v^2 \to \mu_4$ the dispersion relation for flexural waves may be written to leading order by making use of equation (4.14). Inserting the appropriate expansion for ρv^2 from (4.11) yields

$$\hat{q}_1 G(0, \mu_4)\psi(q_2^{(0)}, q_3^{(0)}, \mu_4)(\bar{q}_2^{(0)} - \bar{q}_3^{(0)}) \tan k\hat{q}h$$
$$= q_2^{(0)} G(q_2^{(0)}, \mu_4)\psi(0, q_3^{(0)}, \mu_4)\bar{q}_3^{(0)} - q_3^{(0)} G(q_3^{(0)}, \mu_4)\psi(0, q_2^{(0)}, \mu_4)\bar{q}_2^{(0)} + O(\hat{q}_1^2), \quad (4.15)$$

where $q_1 = i\hat{q}_1$, $q_2^{(0)} = \sqrt{\bar{q}_2^{(0)}}$ and $q_3^{(0)} = \sqrt{\bar{q}_3^{(0)}}$ are the order 1 parts of q_2 and q_3, and $G(0, \mu_4)$ is the order 1 quantity associated with $G(q_1)$, such that

$$G(q_1, \rho v^2) = G(0, \mu_4) + O(\hat{q}_1)^2,$$

and so on for $G(q_2^{(0)}, \mu_4)$, etc. It is inferred from equation (4.15) that $\hat{q}_1 \tan k\hat{q}_1 h \sim O(1)$, implying that as $kh \to \infty$, $\tan k\hat{q}_1 h \to \infty$ and $\hat{q}_1 \to 0$, thus

$$\hat{q}_1 = \left(n + \frac{1}{2}\right)\frac{\pi}{kh} + O(kh)^{-2}. \quad (4.16)$$

By making use of the appropriate expansion for ρv^2 in equation (4.11) a second order expansion to the phase speed is obtained, namely

$$\rho v_n^2 = \mu_4 - \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{(kh)^2} \frac{\mu_4(\mu_4 - \mu_2) + \mu_3}{\mu_4 - \mu_5} + O(kh)^{-3}, \qquad n = 0, 1, 2, 3, \dots \quad . \quad (4.17)$$

A higher order expansion for the phase speed is obtained by setting

$$\hat{q}_1 \approx \left(n + \frac{1}{2}\right)\frac{\pi}{kh} + \frac{\phi}{(kh)^2} + O(kh)^{-3}, \quad (4.18)$$

from which we infer

$$\tan k\hat{q}h = \frac{-kh}{\phi} + O(kh)^{-1}. \quad (4.19)$$

Inserting the expansions shown in equations (4.18) and (4.19) into the dispersion relation (4.15), and equating like powers of kh, yields

$$\phi = \frac{G(0, \mu_4)\psi(q_2^{(0)}, q_3^{(0)}, \mu_4)(\bar{q}_2^{(0)} - \bar{q}_3^{(0)})}{q_2^{(0)} q_3^{(0)}\{q_2^{(0)} G(q_3^{(0)}, \mu_4)\psi(0, q_2^{(0)}, \mu_4) - q_3^{(0)} G(q_2^{(0)}, \mu_4)\psi(0, q_3^{(0)}, \mu_4)\}}\left(n + \frac{1}{2}\right)\pi. \quad (4.20)$$

Finally, a third order approximation to the phase speed is obtained by making use of equation (4.20) in (4.11), thus

$$\rho v_n^2 = \mu_4 - \frac{\mu_4(\mu_4 - \mu_2) + \mu_3}{\mu_4 - \mu_5}\left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{(kh)^2}\left\{1 + \frac{2}{kh}\hat{\phi}\right\} + O(kh)^{-4},$$
$$n = 0, 1, 2, 3\dots, \quad (4.21)$$

where $\hat{\phi} = \frac{\phi}{(n+1/2)\pi}$.

(ii) Limit $\rho v^2 \to \mu_5$

We have mentioned that in the limit $\rho v^2 \to \mu_5$ the leading order term of $\psi(q_1, q_2, \rho v^2)$ and $\psi(q_1, q_3, \rho v^2)$ changes. It may be readily shown from the definition of $\psi(a, b, \rho v^2)$ that the leading order term of $\psi(q_1, q_2, \rho v^2)$ and $\psi(q_1, q_3, \rho v^2)$ are now order $O(\hat{q}_1^2)$, whilst the leading order term of $\psi(q_2, q_3, \rho v^2)$ remains order 1. The leading order terms of $\psi(q_1, q_2, \rho v^2)$ and $\psi(q_1, q_3, \rho v^2)$ are obtained by inserting the expansion for the phase speed (4.12), appropriate to this case, into the definition of $\psi(a, b, \rho v^2)$ given immediately below equation (2.28). If this is done we obtain

$$\psi(q_1, q_2, \rho v^2) = -\hat{q}_1^2 \Gamma(q_2^{(0)}) + O(\hat{q}_1^4), \qquad \psi(q_1, q_3, \rho v^2) = -\hat{q}_1^2 \Gamma(q_3^{(0)}) + O(\hat{q}_1^4), \tag{4.22}$$

where $\Gamma(q)$ is defined to be

$$\begin{aligned}\Gamma(q) = \gamma_{21}\gamma_{23}q^2 \Big\{ &(\gamma_{23} - \gamma_{21})(\gamma_{21}\cos^2\theta + \gamma_{23}\sin^2\theta + \mu_5) \\ &+ \gamma_{23}\big(\epsilon \sin^2\theta - \gamma_{31}\cos^2\theta\big) - \gamma_{21}\big(\delta\cos^2\theta - \gamma_{13}\sin^2\theta\big)\Big\} \\ + \frac{\mu_5(\mu_5 - \mu_2) + \mu_3}{\mu_5 - \mu_4} \Big\{ &\gamma_{21}\gamma_{23}(\gamma_{23} - \gamma_{21})q^2 - \gamma_{21}(\gamma_{23} - \sigma_2)\big(\delta\cos^2\theta - \gamma_{13}\sin^2\theta + \mu_5\big) \\ &+ \gamma_{23}(\gamma_{21} - \sigma_2)\big(\epsilon\sin^2\theta - \gamma_{31}\cos^2\theta + \mu_5\big)\Big\}.\end{aligned}$$

By using equation (4.22), in conjunction with the expansions for the phase speed (4.12), the dispersion relation (2.29) may be written to leading order in the form

$$\begin{aligned}&\hat{q}_1 G(0, \mu_5)\psi(q_2^{(0)}, q_3^{(0)}, \mu_5)(\bar{q}_2^{(0)} - \bar{q}_3^{(0)}) \tan k\hat{q}_1 h \\ &\qquad = \hat{q}_1^2 \big\{\bar{q}_2^{(0)} q_3^{(0)} G(q_3^{(0)}, \mu_5)\Gamma(q_2^{(0)}) - q_2^{(0)}\bar{q}_3^{(0)} G(q_2^{(0)}, \mu_5)\Gamma(q_3)\big\} + O(q_1^3),\end{aligned} \tag{4.23}$$

from which it is deduced that $O(1)\tan k\hat{q}_1 h \sim O(\hat{q}_1)$, implying that as $\hat{q}_1 \to 0$ $\tan k\hat{q}_1 h \to 0$, and therefore

$$\hat{q}_1 = \frac{n\pi}{kh} + O(kh)^{-2}. \tag{4.24}$$

A second order expansion of the phase speed is obtained by making use of equation (4.24) in (4.12), namely

$$\rho v^2 = \mu_5 - \Big(\frac{n\pi}{kh}\Big)^2 \frac{\mu_5(\mu_5 - \mu_2) + \mu_3}{\mu_5 - \mu_4} + O(kh)^{-3}. \tag{4.25}$$

We then seek a higher order expansion of the phase speed by setting

$$\hat{q}_1 = \frac{n\pi}{kh} + \frac{\phi_2}{(kh)^2} + O(kh)^{-3}, \qquad \tan k\hat{q}_1 h = \frac{\phi_2}{kh} + O(kh)^{-3} \quad . \tag{4.26}$$

Inserting the expansions shown above in equation (4.26) into the leading order term of the dispersion relation (4.23), and on comparing like powers of kh, it is deduced that

$$\phi_2 = \frac{q_2^{(0)} q_3^{(0)} \big\{q_2^{(0)} G(q_3^{(0)}, \mu_5)\Gamma(q_2^{(0)}) - q_3^{(0)} G(q_2^{(0)}, \mu_5)\Gamma(q_3^{(0)})\big\}}{G(0, \mu_4)\psi(q_2^{(0)}, q_3^{(0)}, \mu_5)(\bar{q}_2^{(0)} - \bar{q}_3^{(0)})} n\pi. \tag{4.27}$$

A third order expansion of the phase speed is then derived by inserting equation (4.27) into (4.12), giving

$$\rho v^2 = \mu_5 - \frac{\mu_5(\mu_5-\mu_2)+\mu_3}{\mu_5-\mu_4}\left(\frac{n\pi}{kh}\right)^2\left(1+\frac{2}{kh}\hat{\phi}_2\right) + O(kh)^{-4}, \qquad n = 1,2,3,\dots \quad . \tag{4.28}$$

where $\hat{\phi}_2 = \phi_2/\pi$.

The asymptotic expansions derived in equations (4.21) and (4.28) are compared graphically with numerical solutions in figures (11) and (12), respectively. The numerical solutions have been generated using the material parameters in table (1). In figure (11) $\theta = 75^o$ and the limiting wave speed of the harmonics is $\rho v^2 = \mu_4$. Asymptotic expansions are superimposed on numerical solutions for the first four harmonics and indicate a reasonable degree of accuracy. For figure (12) the angle of propagation is $\theta = 15^o$ and the limiting wave speed of the harmonics is therefore $\rho v^2 = \mu_5$. We again obtain a reasonable approximation to the phase speed, especially for the first and second harmonics. As the harmonic number increases we find that a larger value of kh is required before we obtain a reasonable approximation. This is to be expected as within the asymptotic expansions (4.21) and (4.28) we require $n \ll kh$.

[Figures 11 and 12 about here.]

4.3.2 Extensional waves

Now consider the corresponding limits for extensional waves in the single layer. Again, we shall investigate the two limits $\rho v^2 \to \mu_4$ and $\rho v^2 \to \mu_5$ separately.

(i) Limit $\rho v^2 \to \mu_4$

The dispersion relation for extensional waves (2.32) may be represented in the high wave number regime to leading order by making use of the expansions in equations (4.11), (4.12) and (4.14), thus

$$\hat{q}_1 G(0,\mu_4)\psi(q_1^{(0)}, q_2^{(0)}, \mu_4)(\bar{q}_2^{(0)} - \bar{q}_3^{(0)}) = \\ \left\{ q_2^{(0)} G(q_2^{(0)}, \mu_4)\psi(0, q_3^{(0)}, \mu_4)\bar{q}_3^{(0)} - q_3^{(0)} G(q_3^{(0)}, \mu_4)\psi(0, q_2^{(0)}, \mu_4)\bar{q}_2^{(0)} \right\} \tan k\hat{q}h + O(q_1^2), \tag{4.29}$$

from which is deduced that $O(\hat{q}_1) \sim O(1)\tan k\hat{q}_1 h$. This implies that $\tan k\hat{q}_1 h \to 0$ as $\hat{q}_1 \to 0$, and

$$\hat{q}_1 = \frac{n\pi}{kh} + O(kh)^{-2}. \tag{4.30}$$

A second order expansion of the phase speed is obtained by inserting equation (4.30) into (4.12), namely

$$\rho v^2 = \mu_4 - \left(\frac{n\pi}{kh}\right)^2 \frac{\mu_4(\mu_4-\mu_2)+\mu_3}{\mu_4-\mu_5} + O(kh)^{-3}. \tag{4.31}$$

A higher order expansion is again sought by setting

$$\hat{q}_1 = \frac{n\pi}{kh} + \frac{\phi_3}{(kh)^2} + O(kh)^{-2}, \qquad \tan k\hat{q}_1 h = \frac{\phi_3}{kh^2} + O(kh)^{-3} \quad . \tag{4.32}$$

On inserting equation (4.32) into the leading order term of the dispersion relation (4.29) it is found that

$$\phi_3 = \frac{G(0,\mu_4)\psi(q_2^{(0)},q_3^{(0)},\mu_4)(\bar{q}_2^{(0)} - \bar{q}_3^{(0)})}{q_2^{(0)}q_3^{(0)}\{q_2^{(0)}G(q_3^{(0)},\mu_4)\psi(0,q_2^{(0)},\mu_4) - q_3^{(0)}G(q_2^{(0)},\mu_4)\psi(0,q_3^{(0)},\mu_4)\}} n\pi, \tag{4.33}$$

where it is noted that $\phi_3/n\pi = \hat{\phi}$. The appropriate expansion of the phase speed is then obtained by inserting equation (4.33), in conjunction with (4.32), into equation (4.13), thus

$$\rho v_n^2 = \mu_4 - \frac{\mu_4(\mu_4 - \mu_2) + \mu_3}{\mu_4 - \mu_5}\left(\frac{n\pi}{kh}\right)^2 \left\{1 + \frac{2}{kh}\hat{\phi}\right\} + O(kh)^{-4}, \qquad n = 1,2,3\dots \quad . \tag{4.34}$$

(ii) Limit $\rho v^2 \to \mu_5$

The leading order term of the extensional dispersion relation as $\rho v^2 \to \mu_5$ is obtained by making use of equations (4.13), (4.14) and (4.22) in the dispersion relation (2.32), to obtain

$$\hat{q}_1 G(0,\mu_5)\psi(q_2^{(0)},q_3^{(0)},\mu_5)(\bar{q}_2^{(0)} - \bar{q}_3^{(0)}) = \hat{q}_1^2\{\bar{q}_2^{(0)}q_3^{(0)}G(q_3^{(0)},\mu_5)\Gamma(q_2^{(0)}) - q_2^{(0)}\bar{q}_3^{(0)}G(q_2^{(0)},\mu_5)\Gamma(q_3)\}\tan k\hat{q}_1 h + O(q_1^3). \tag{4.35}$$

From equation (4.35) we deduce that $O(1) \sim O(\hat{q}_1)\tan k\hat{q}_1 h$, implying that $\tan k\hat{q}_1 h \to \infty$ as $\hat{q}_1 \to 0$, and therefore

$$\hat{q}_1 = \left(n + \frac{1}{2}\right)\frac{\pi}{kh} + O(kh)^{-2}. \tag{4.36}$$

A second order expansion of the phase speed is obtained by inserting equation (4.36) into (4.13), giving

$$\rho v_n^2 = \mu_5 - \left(n + \frac{1}{2}\right)^2 \frac{\pi^2}{(kh)^2}\frac{\mu_5(\mu_5 - \mu_2) + \mu_3}{\mu_5 - \mu_4} + O(kh)^{-3}, \qquad n = 0,1,2,\dots \quad . \tag{4.37}$$

A higher order expansion is again sought by setting

$$\hat{q}_1 = \left(n + \frac{1}{2}\right)\frac{\pi}{kh} + \frac{\phi_4}{(kh)^2} + O(kh)^{-3}, \qquad \tan k\hat{q}_1 h = \frac{-kh}{\phi_4} + O(kh)^{-2} \quad . \tag{4.38}$$

Inserting the expansions indicated above into (4.32), and on comparing like powers of kh it is deduced that

$$\phi_4 = \frac{q_2^{(0)}q_3^{(0)}\{q_2^{(0)}G(q_3^{(0)},\mu_5)\Gamma(q_2^{(0)}) - q_3^{(0)}G(q_2^{(0)},\mu_5)\Gamma(q_3^{(0)})\}}{G(0,\mu_5)\psi(q_2^{(0)},q_3^{(0)},\mu_5)(\bar{q}_2^{(0)} - \bar{q}_3^{(0)})}\left(n + \frac{1}{2}\right)\pi, \tag{4.39}$$

where it is again noted that $\phi_4/(n+0.5)\pi = \hat{\phi}_2$. Finally, inserting equation (4.39) into (4.12) yields a third order expansion of the phase speed, namely

$$\rho v_n^2 = \mu_5 - \frac{\mu_5(\mu_5-\mu_2)+\mu_3}{\mu_5-\mu_4}\left(n+\frac{1}{2}\right)^2 \frac{\pi^2}{(kh)^2}\left\{1+\frac{2}{kh}\hat{\phi}_2\right\} + O(kh)^{-4},$$

$$n = 0, 1, 2, \ldots \quad . \qquad (4.40)$$

The asymptotic expansions obtained in equations (4.34) and (4.40) are compared graphically with numerical solutions in figures (13) and (14), respectively. The numerical solutions are again generated for the material parameters indicated in table (1). The angle of propagation in figure (13) is $\theta = 75^o$, which corresponds to limiting wave speed of $\rho v^2 = \mu_4$, while in figure (14) $\theta = 15^o$ and the limiting wave speed is $\rho v^2 = \mu_5$. In both cases we obtain a good approximation to the phase speed.

[Figures 13 and 14 about here.]

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